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J. W. L. GLAISHER, M.A., F.R.S., President, in the Chair.

Commander James Llewellyn Heane, F.R.G.S., Alte Burg, bei Gross Berkel, Hanover,

was balloted for and duly elected a Fellow of the Society.

On Delaunay's Method for Calculating Terms of Long Period in the Motion of the Moon. By E. Neison.

In the *Monthly Notices* for March 1878 (vol. xxxviii. pp. 268–279), I drew attention to an important discrepancy between the theoretical investigations of Hansen and Delaunay upon which they had founded their evaluations of the terms of long period in the motion of the moon arising from the disturbing action of the planet *Venus*; and I pointed out that it seemed very doubtful whether Delaunay's investigations were really complete, for there appeared to be strong evidence that he had neglected the very terms from which Hansen had derived the large values found by him for the *Venus* terms.

It is well known that Hansen himself never admitted the accuracy of these investigations of Delaunay, but always maintained that they were erroneous owing to some imperfection in the method, though he does not seem ever to have definitely pointed out the manner in which he believed it to be imperfect. It is known that he was in communication with Leverrier, and furnished him with reasons sufficiently strong to bring Leverrier to the same opinion. Delaunay's method, however, differs completely from those employed in their researches by both Hansen and Leverrier, and though its fundamental theoretical principle is beautifully simple and rigidly accurate, there are involved a very complex series of transformations and approximations before

it is applicable to the actual calculation of the lunar theory. And a study of their published papers has led me to the view that though both Hansen and Leverrier had convinced themselves that Delaunay's method had led him to imperfect results, neither had considered it necessary to obtain that mastery over its complex transformations and their involved approximations that would alone have enabled them to indicate the exact point where it became imperfect.

I reproduce here from my Paper of March 1878 the reasons which led me to the opinion that Delaunay's method is incomplete:

"In the *Memoirs of the Royal Astronomical Society*, vol. xvi. Hansen has published the only existing details of his method of computing these inequalities, and on page 476 there occurs the following passage. Speaking of the term whose argument is denoted by him by $(g-18g''+16g')$, and whose value as far as the first power of the disturbing forces he had just computed, Hansen says: 'Cependant le cube des forces perturbatrices ou plutôt le produit du carré et du cube de la force perturbatrice du soleil par la force perturbatrice de *Venus*, ajoute une quantité sensible à ce coefficient, et rend en même temps sensible le coefficient de l'argument $(8g''-13g')$, qui, en tenant seulement compte de le première puissance de la force perturbatrice, serait insensible dans le mouvement de la lune.' Delaunay appears to me to have entirely overlooked the real bearing of this important statement, for I think it clearly defines from whence Hansen obtained the large value for the coefficient of the argument $(8g''-13g')$, which was found to be insensible by Delaunay.

"The direct action of the planet *Venus* in producing perturbations in the motion of the Moon may be analytically represented by a particular disturbing function made to form part of the complete disturbing function which represents the action of every disturbing body on the Moon. This particular disturbing function depends on the undisturbed geocentric co-ordinates of the Moon and *Venus* and on the complete perturbations of the Moon. By neglecting the second and higher powers of the mass of *Venus*, this particular disturbing function is developed in terms of the heliocentric elliptic co-ordinates of *Venus* and the Earth, the geocentric elliptic co-ordinates of the Moon, and of the different powers of the perturbations in the motion of the Moon which are produced by the disturbing action of the Sun. Delaunay has exclusively confined himself to this portion of the complete disturbing force when finding the direct action of *Venus* upon the Moon, and as far as the first power of the disturbing forces, it is evident that this is the only portion which need be considered. As far as the first power of the disturbing forces, Delaunay finds a value for the term $(g-18g''+16g')$ which is sensibly identical with that found by Hansen under the like conditions. For the term $(8g''-13g')$, under these conditions, Delaunay finds its coefficient to be insensible. In this also

he agrees with Hansen, as the paragraph which has been quoted will show. To this order Delaunay's investigations must be complete, and to this order both Hansen and Delaunay have obtained identical results.

"When Delaunay takes into consideration the effect of the different powers of the perturbations in the motion of the Moon arising from the disturbing action of the Sun, in all cases he considers the effect of the first power of these perturbations, and in one particular term he includes the second power of these perturbations. Delaunay shows that these terms do not materially augment the value of the two terms of long period. Granting that these results are perfectly accurate—and there can be no real doubt about their accuracy—is there any real discrepancy between them and those which Hansen obtained? It cannot be said that there is; for in the paragraph which has been quoted Hansen explicitly states that the coefficient of the term $(8g''-13g')$ is insensible *unless* there be included the terms depending on the product of the first power of the disturbing force of *Venus* with the *square* and *cube* of the disturbing force of the Sun. *But Delaunay has entirely omitted the terms of this order in his investigation.* Delaunay himself does not seem to have noticed this discrepancy; partly because he seems to call the first power of the disturbing force of *Venus*, also the first power of the disturbing force of the Sun. And his investigation being performed with the aid of his new and elegant method of attacking the lunar theory, in this portion they are not easily followed by anyone not perfectly familiar with this most laborious method. Had this not been so, it is probable that attention would have been directed before now to this discrepancy.

"There is little difficulty in ascertaining what probably were Delaunay's reasons for omitting these terms. It is obvious that the terms neglected will be smaller than those which he has retained, and which depend upon the first power of the disturbing force of the Sun. But these last are very small, so that there could be no valid reason for including the others which were still smaller. Delaunay, therefore, was perfectly justified in omitting these terms. But this very circumstance is enough to suggest the inquiry whether Hansen was really referring to these terms, or whether in his remarks he does not refer to some entirely distinct class of terms whose existence has been overlooked by Delaunay. Is there, then, a second class of terms to which Hansen's remarks may really apply?"

In my Paper I then proceed to show that on developing the differential equations in the ordinary manner, there do arise a class of terms which correspond exactly with Hansen's description, namely the terms which express the direct action of the Sun upon the perturbations in the motion of the Moon which are produced by the direct action of the planet; and show, further, that it is to these terms, and these only, to which Hansen could have referred in the paragraph already quoted, as

the portion of his investigation from which he obtained the large value of the coefficient with the argument $(8g''-13g')$. As it seemed certain that Delaunay had entirely neglected these terms, in this difference appeared the origin of the discrepancy between the results obtained by the two investigators.

The evidence thus adduced seemed to me to be conclusive, so I have on several occasions referred to this Paper of mine as establishing the incompleteness of Delaunay's evaluation of these terms. Moreover, I have since verified the opinion that it was to these terms that Hansen refers, by proving, by the actual development of the equations, that Hansen's method does lead to the formation of these terms, and that they do render sensible the coefficient of this *Venus* term.

Lately there has appeared good reason for believing that this evidence is not deemed by others to be nearly so conclusive as it appears to me, and that more than one distinguished astronomer remained unconvinced of the insufficiency of this method of Delaunay's.

Moreover, of late, Delaunay's method of calculating terms of long period has been applied by other astronomers to the determination of the values of further terms of this kind, only with the result of finding the same discrepancy between the values so obtained and those yielded by other methods. In each case the value obtained by following Delaunay's method, which may be called the method of transformed elements, yields only a small fraction of the result obtained by the methods of perturbed polar co-ordinates or of the direct variation of elliptic elements. Thus, the discrepancy to which Prof. Adams and Mr. Hind, as well as Prof. Newcomb, have directed attention, between the results obtained by M. Gogou for the term due to the action of *Mars* with the argument $(l-24l''+20l''')$, and by Prof. G. W. Hill for the Jovian Perigee term with the argument $(2\theta_{iv}-2a)$, and those previously obtained by myself by the method of perturbed co-ordinates, forms merely an exact repetition of the discordance between the values for the *Venus* term $(8l'-13l'')$ found by Prof. Hansen and M. Delaunay by the same difference of method.

The question involved, therefore, is clearly not a mere difference in computation, such as might be ascribed to some oversight or error, but is a fundamental difference in principle between the results derived from the method of transformed elements, as applied to these terms by M. Delaunay, M. Gogou, and Prof. Hill, and the results of the methods of perturbed co-ordinates or perturbed elliptic elements as employed by Prof. Hansen and myself.

This statement places the matter on a clear footing; it remains to trace to its source this fundamental difference in principle between the two results.

I would place the question in this form. *Does the method employed by M. Delaunay yield the complete value of these coefficients?*

With Hansen, I say, No!

Let me clearly define the exact nature of my dissent from the method employed by Delaunay for the investigation of these terms of long period. I do not question the principle on which it is founded. I admit as freely as I admire the beautiful simplicity of the principle on which Delaunay has founded his development of the Lunar Theory. I admit the accuracy of the results embodied in the two great volumes forming tomes xxviii and xxix of the *Mémoires de l'Académie des Sciences.* As there applied it yields perfect results. My contention is that, in applying this method to the calculation of the perturbations produced by the disturbing action of the planets, he has omitted to consider a portion of the complete disturbing function which must be taken into consideration in order to obtain the complete value of these perturbations.

Consider first the development of the disturbing function which is to serve as the basis of the investigation.

Let R denote the disturbing function when the elements of the orbit of the Moon have their instantaneous elliptic values denoted by

$$x\ y\ z \quad X\ Y\ Z.$$

Further, let the general term of R be denoted by

$$R\cos w = (A)\,\epsilon^{Fx+Gy+Hz}\,.\,\cos\{(\lambda)+fX+gY+hZ\} \qquad (1)$$

where (A) and (λ) are supposed to be independent of the elements of the disturbed orbit, and to be absolutely constant, therefore, as far as concerns any variations in these elements.

Suppose R to denote the value of R, when the elements of the disturbed orbit have their constant mean values, so that R will be the same function of the mean elements denoted by

$$x\ y\ z \quad X\ Y\ Z,$$

as R is of the instantaneous elements

$$x\ y\ z \quad X\ Y\ Z.$$

Hence the general term of R may be written in the form

$$R\cos\omega = (A)\,\epsilon^{Fx+Gy+Hz}\,.\,\cos\{(\lambda)+fX+gY+hZ\} \qquad (2)$$

Let each element of the orbit be regarded as the sum of the mean constant value, and of a series of periodical functions which collectively represent the periodical perturbations of the element, so that, for instance,

$$X = x+\delta x = x+\delta_1 x+\delta_2 x+\delta_3 x+\ \ldots\ = x+\Sigma\,\delta_m x \qquad (3)$$

Then R is to be developed in terms of R, and of these variables which represent the periodical perturbations in the elements of the disturbed orbit.

Let u be taken as the general representative of the similar

elements which have been denoted by x y z, and let I be taken for the general representative of the corresponding coefficients F G H. Similarly, let U be supposed to represent the similar elements which have been denoted by X Y Z, and i to represent the corresponding coefficients $f\ g\ h$. Further, suppose

$$\begin{aligned} \mathrm{u} &= u + \delta u \quad = u + \Sigma\delta_m u \\ \mathrm{U} &= U + \delta U \quad = U + \Sigma\delta_m U \end{aligned} \tag{4}$$

where, as before, $\delta u\ \delta U$ represent the periodic perturbations of the elements whose mean constant values are denoted by $u\ U$.

To effect the desired development, assume that

$$\begin{aligned} \delta_m u &= \Sigma j^m . \frac{\mathrm{I}}{\omega} . R^m \cos\omega \\ \delta_m U &= \Sigma J^m . \frac{\mathrm{I}}{\omega} . R^m \sin\omega \end{aligned} \tag{5}$$

where on the right-hand side the affix m indicates the number of accents to be attached to the letters $j\ J\ R$.

Next, by successive differentiation

$$\begin{aligned} \frac{d^s R}{d u^s} &= \Sigma I^s R \cos\omega \\ \frac{d^s R}{d U^s} &= \Sigma i^s \left\{ \cos s\frac{\pi}{2} . R\cos\omega - \sin s\frac{\pi}{2} . R\sin\omega \right\} \end{aligned} \tag{6}$$

Put

$$\begin{aligned} \mathrm{D}_u &= \delta u . \frac{d}{d u} \\ \mathrm{D}_U &= \delta U . \frac{d}{d U} \end{aligned}$$

then

$$\begin{aligned} \mathrm{R} &= R + \delta R = \{\epsilon^{\Sigma(\mathrm{D}_u + \mathrm{D}_U)}\} R \\ &= \{\Sigma(1 + \mathrm{D}_u + \tfrac{1}{1\cdot 2}[\mathrm{D}_u]^2 + \ldots .)(1 + \mathrm{D}_U + \tfrac{1}{1\cdot 2}[\mathrm{D}_U]^2 + \ldots .)\} R \end{aligned} \tag{7}$$

For conciseness put

$$\begin{aligned} [R_s^m \quad (I_r j_s^m)\frac{\mathrm{I}}{\omega_s}]^k &= (I_r)^k (j_{s_1}^m R_{s_1}^m \frac{\mathrm{I}}{\omega_{s_1}} + \ldots . + j_{s_k}^m R_{s_k}^m \frac{\mathrm{I}}{\omega_{s_k}})^k \\ [R_s^m (q\, i_r\, J_s^m)\frac{\mathrm{I}}{\omega_s}]^k &= (q' q'' \ldots q^k)(i_r)^k (J_{s_1}^m R_{s_1}^m \frac{\mathrm{I}}{\omega_{s_1}} + \ldots . + J_{s_k}^m R_{s_k}^m \frac{\mathrm{I}}{\omega_{s_k}})^k \end{aligned} \tag{8}$$

and further, let Σ_m denote a summation with regard to the affix m, so that, for example,

$$\begin{aligned} [\Sigma_m R_s^m (I_r j_s^m)\frac{\mathrm{I}}{\omega_s}]^k = (I_r)^k \{ &[j'_{s_1} R'_{s_1} + \ldots . + j_{s_1}^m R_{s_1}^m + \ldots .]\frac{\mathrm{I}}{\omega_{s_1}} + \\ &\ldots . + [j_{s_k} R_{s_k} + \ldots . + j_{s_k}^m R_{s_k}^m + \ldots .]\frac{\mathrm{I}}{\omega_{s_k}} \}^k \end{aligned} \tag{9}$$

In these expressions the suffixes $r, s\, s_1\, s_2 \ldots . s_k$ are supposed to be numbers employed to distinguish different coefficients of the classes $R\, I\, i\, J\, j \ldots .$ belonging to an argument. Thus each of these classes of co-efficient belonging to an argument ω_{s_k} distinguished from other arguments by carrying the subscript number denoted by $_{s_k}$, will be distinguished from other similar coefficients by being written in the form

$$R_{s_k} . I_{s_k}\, i_{s_k}\, J_{s_k}\, j_{s_k} \ldots .$$

Let the prefix Σ_u attached to any function of the general coefficients $I\, i \ldots .$ indicate that they take in succession every value included under the general term, so that, for example,

$$\Sigma_u (I_{s_1} i_{s_2} - q\, i_{s_1} I_{s_2}) \epsilon^{-u} = (F_{s_1} f_{s_2} - q\, f_{s_1} F_{s_2}) \epsilon^{-x} + (G_{s_1} g_{s_2} - q\, g_{s_1} G_{s_2}) \epsilon^{-y} + (H_{s_1} h_{s_2} - q\, h_{s_1} H_{s_2}) \epsilon^{-z} \qquad (10$$

If, however, this sign be not prefixed, then it is to be understood that the coefficient takes only the particular value corresponding to the particular value assigned to u. Thus, in the general expression for the perturbations of u, suppose u to represent x, then

$$(I_{s_1} i_{s_2} - q\, i_{s_1} I_{s_2}) \epsilon^{-u} = (F_{s_1} f_{s_2} - q\, f_{s_1} F_{s_2}) \epsilon^{-x}$$

These conventions have been introduced solely for the purpose of conciseness, as they obviate the necessity of writing out all the coefficients.

From the values assigned to the symbol D the following expressions can be derived :—

$$[\mathrm{D}_u]^k R \cos \omega = \left\{ \tfrac{1}{2} \Sigma_m R_s^m [\Sigma_u (I_r j_s^m)] \frac{\mathrm{I}}{\omega_s} \right\}^k \cos (\omega_r + q' \omega_{s_1} + q'' \omega_{s_2} + \ldots + q^k \omega_{s_k})$$

$$[\mathrm{D}_U]^k R \cos \omega = \left\{ \tfrac{1}{2} \Sigma_m R_s^m [\Sigma_u (q\, i_r J_s^m)] \frac{\mathrm{I}}{\omega_s} \right\}^k \cos (\omega_r + q' \omega_{s_1} + q'' \omega_{s_2} + \ldots + q^k \omega_{s_k})$$

Hence

$$\{[\mathrm{D}_u]^{k'} [\mathrm{D}_U]^{k''}\} R \cos \omega = \left\{ \tfrac{1}{2} \Sigma_m R_s^m [\Sigma_u (I_r j_s^m)] \frac{\mathrm{I}}{\omega_s} \right\}^{k'}$$

$$\times \left\{ \tfrac{1}{2} \Sigma_m R_s^m [\Sigma_u (q\, i J_s^m)] \frac{\mathrm{I}}{\omega_s} \right\}^{k''} \cos (\omega_r + q' \omega_{s_1} + q'' \omega_{s_2} + \ldots . + q^{k'+k'} \omega_{sk'+k''})$$

In these expressions the coefficients $q'\ q''\ q''' \ldots .$ only take the values plus or minus unity, and have been introduced for the purpose of more clearly showing the changes which occur in the coefficients where changes are introduced into the arguments. Thus, by giving in succession the values plus or minus unity to the coefficients $q'\ q''\ q''' \ldots . q^k$ in the argument denoted by

$$\omega = \omega_r + q' \omega_{s_1} + q'' \omega_{s_2} + \ldots . + q^k \omega_{s_k}$$

it will take in succession every value which can arise from the combination of the argument $\omega_{s_1} \ldots \omega_{s_k}$ with the argument ω_r, and by giving the same values to $q' \ldots q_k$ in the preceding expressions for the coefficients attached to this argument, there will be obtained the correct value of the corresponding coefficients.

By substituting these values of the functions denoted by the symbol D in the expression for R there will be obtained the complete development of the general term of the disturbing function in the form

$$\begin{aligned} R\cos w = R\cos\omega + & \left\{\tfrac{1}{2}\Sigma_m R^m_{s_1}[\Sigma_u(I_r j^m_{s_1} + q\, i_r J^m_{s_1})]\,\frac{I}{\omega_{s_1}}\right\} R_r\cos(\omega_r + q'\omega_{s_1}) \\ & + \ldots\ldots \\ & + \frac{I}{k}\left\{\tfrac{1}{2}\Sigma_m R^m_s[\Sigma_u(I_r j^m_s + q' i_r J^m_s)]\,\frac{I}{\omega_s}\right\}^k R_r\cos(\omega_r + \Sigma^k q\omega_s) \\ & + \ldots\ldots \end{aligned} \tag{12}$$

This expression will give, in turn, the complete development of R in terms of R and the periodical perturbations.

It remains to obtain R in an explicit form.

Let ρ denote the geocentric distance of the disturbing body; H denote the geocentric angular distance between the disturbing body and the Moon; and rVλ denote the geocentric radius vector, true longitude and latitude of the Moon in its disturbed orbit. Then, putting M for the mass of the disturbing body, the function which expresses its disturbing action on the Moon will be

$$R = M\left\{\frac{r^2}{\rho^3}\left(\tfrac{3}{2}\cos^2 H - \tfrac{1}{2}\right) + \ldots\ldots\right\} \tag{13}$$

Suppose the disturbing body to be a planet moving in an orbit so slightly inclined to the ecliptic that its latitude may be regarded as insensible. Put $r_1 V_1$ for the heliocentric radius vector and longitude of the planet, and $r_\circ\ V_\circ$ for the heliocentric radius vector and longitude of the Earth; then, as

$$\begin{aligned} \rho^2 &= r_1^2 + r_\circ^2 - 2r_1 r_\circ\cos(V_1 - V_\circ) \\ \cos H &= \cos\lambda\left\{\frac{r_1}{\rho}\cos(V - V_1) - \frac{r_\circ}{\rho}\cos(V - V_\circ)\right\} \end{aligned}$$

the disturbing function expressing the direct action of a planet upon the Moon may be written as

$$\begin{aligned} R_1 = M_1\Big\{ & \tfrac{1}{4}\cdot\frac{r^2}{\rho^3}(1 - 3\sin^2\lambda) + \tfrac{3}{4}\cdot\frac{r^2}{\rho^5}(1 - \sin^2\lambda)[r_1^2\cos(2V - 2V_1) \\ & + r_\circ^2\cos(2V - 2V_\circ) + 2r_1 r_\circ\cos(2V - V_1 - V^\circ)] + \ldots\ldots\Big\} \end{aligned}$$

The ordinary disturbing function of the lunar theory which represents the direct disturbing action of the Sun can be obtained

from the preceding by making r_1 equal to zero. It can be written, therefore, as

$$R_0 = M_0 \left\{ \tfrac{1}{4} \cdot \frac{r^2}{r_0^3} (1 - 3\sin^2\lambda) + \tfrac{3}{4} \cdot \frac{r^2}{r_0^3} (1 - \sin^2\lambda) \cos(2V - 2V_0) + \dots \right\} \quad (14)$$

These form the quantities to be developed in the manner already indicated.

Suppose the method of investigation adopted be that of perturbed co-ordinates, similar to that employed in my general method of treating the lunar theory (*Memoirs*, vol. xliv. 1878), so that R is to be developed in terms of the elliptic co-ordinates and their perturbations.

To simplify the expressions, suppose the minor effects arising from the perturbations in the latitude of the Moon to be disregarded, so that the perturbed co-ordinates of the Moon may be supposed reduced to the radius vector and longitude.

Further, let

v = logarithm of the radius vector of the disturbed orbit.

V = the longitude in the disturbed orbit.

And suppose

$$\begin{aligned}
A_1 &= \tfrac{1}{4} \cdot M_0 \cdot \frac{1}{r_0^3} & \lambda_1 &= 0 \\
A_2 &= \tfrac{3}{4} \cdot M_0 \cdot \frac{1}{r_0^3} & \lambda_2 &= -2V_0 \\
A_3 &= \tfrac{1}{4} \, M_1 \cdot \frac{1}{\rho^3} & \lambda_3 &= 0 \\
A_4 &= \tfrac{3}{4} \, M_1 \cdot \frac{1}{\rho^3} \cdot \frac{r_1^2}{\rho^2} & \lambda_4 &= -2V_1 \\
A_5 &= \tfrac{3}{4} \, M_1 \cdot \frac{1}{\rho^3} \cdot \frac{r_0^2}{\rho^2} & \lambda_5 &= -2V_0 \\
A_6 &= -\tfrac{3}{4} \, M_1 \cdot \frac{1}{\rho^3} \cdot \frac{2r_1 r_0}{\rho^2} & \lambda_6 &= -V_1 - V_0
\end{aligned} \quad (15)$$

Then the preceding expressions for the disturbing function reduce to

$$\begin{aligned}
R_0 &= (A_1)\epsilon^{2v} \cos\{(\lambda_1)\} + (A_2)\epsilon^{2v} \cos\{(\lambda_2) + 2V\} + \dots \\
R_1 &= (A_3)\epsilon^{2v} \cos\{(\lambda_3)\} + (A_4)\epsilon^{2v} \cos\{(\lambda_4) + 2V\} + (A_5)\epsilon^{2v} \cos\{(\lambda_5) + 2V\} \\
&\quad + (A_6)\epsilon^{2v} \cos\{(\lambda_6) + 2V\} + \dots
\end{aligned} \quad (16)$$

Each of these is merely a particular case of the general form (2), and can be developed in the manner already indicated.

For the present let the subscript $_0$ be employed to denote quantities introduced through the disturbing action of the Sun, and the subscript $_1$ to denote quantities introduced through the disturbing action of the planet. Further, as before, suppose R to denote the value of the disturbing function R when the lunar co-ordinates have their undisturbed values. Then it is known

that R can be developed in a series of cosines with arguments which are explicit functions of the time. It may be assumed, therefore, that

$$R = R_0 + R_1 = \Sigma R_0 \cos \beta_0 + \Sigma R_1 \cos \beta_1 = \Sigma R \cos \beta \qquad (17)$$

Let the elliptic values of r v V be denoted by $r\ v\ V$, and suppose the perturbations of these to be expressed in the forms

$$\begin{aligned} \delta v &= \Sigma a_0^m \cos \alpha_0 + \Sigma a_1^m \cos \alpha_1 \\ \delta V &= \Sigma b_0^m \sin \alpha_0 + \Sigma b_1^m \sin \alpha_1 \end{aligned} \qquad (18)$$

Then the above give the values of the general symbols $Ii\,Jj\,\frac{1}{\omega}$, which enter into the expression (12) for the development of the general term of the disturbing function.

Employing the values given above, it is easy to transform the complete development of the disturbing function into the following form, where quantities depending on the square and cube of the disturbing force of the planet have not been written down.

$$\begin{aligned} \mathrm{R} = \quad & \Sigma\, . R_0 A_0^q \cos (\beta_0 + q\alpha_0) \\ & + \Sigma\ R_1 A_0^q \cos (\beta_1 + q\alpha_0) \\ & + \Sigma\ R_0 A_0^q [a_1 + q_1 b_1] \cos (\beta_0 + q\alpha_0 + q_1 \alpha_1) \end{aligned} \qquad (19)$$

where, for convenience, the entire perturbations of the Sun with any given argument have been summed in the form

$$A_0^q \cos \alpha_0 = \{1 + [a_0 + qb_0] + \tfrac{1}{2}[a_0 + qb_0]^2 + \ldots\ldots\} \cos \alpha_0$$

A certain amount of interpretation is involved in the expression for the disturbing function written in the preceding highly condensed form, thus the quantities b_0 become zero for every term whose argument β is explicitly independent of V or of l, the Moon's mean longitude. But for the present purpose this is immaterial, as all three terms of (19) are to be identically interpreted, and when required the need for this interpretation can be obviated by writing each term in a somewhat more extended form.

The preceding expression for R consists of three terms: the first yields the ordinary lunar perturbations depending purely on the disturbing force of the Sun; the second yields the ordinary perturbations due to the direct action of the planet; and the third yields the indirect planetary perturbations in the motion of the Moon due to the action of the Sun on the perturbations introduced into the lunar motions by the direct action of the planet. These may here be termed the perturbations of Classes I., II., and III.

It only remains to substitute this value of the disturbing function in the ordinary differential equations to the lunar coordinates, to enable the perturbations produced by the disturb-

ing action of the planet to be determined with the same exactitude as the ordinary perturbations due to the disturbing action of the Sun.

It is in this manner that I first calculated the value of these perturbations with arguments of long period.

But it is possible to treat the preceding disturbing function under its general form (12) in another and more direct way, that is by the method of disturbed elements. The quantities

$$x\ y\ z \qquad X\ Y\ Z$$

have not been defined, beyond by the conditions implied in the form assigned to the general term (1) of R.

These quantities may be conveniently defined so that they lead to a canonical system of differential equations of the form

$$\begin{aligned} \frac{du'}{dt} &= -\frac{dR}{dU'} \\ \frac{dU'}{dt'} &= -\frac{dR}{du'} \end{aligned} \tag{20}$$

but they may be still more symmetrically employed by slightly departing from the preceding form, and defining them by the condition

$$\begin{aligned} \frac{du}{dt} &= \epsilon^{-u}\frac{dR}{dU} = -\epsilon^{-u}\Sigma iR \sin w \\ \frac{dU}{dt} &= -\epsilon^{-u}\frac{dR}{du} = -\epsilon^{-u}\Sigma IR \cos w \end{aligned} \tag{21}$$

where the system of elements is such that the three elements typified by u are simple exponential functions of the three elliptic constants, the mean distance, eccentricity, and sine of semi-inclination; and the three elements typified by U represent the three semi-periodic elliptic elements, the mean longitude, longitude of perigee, and longitude of node.

The differential equations depend, therefore, merely on particular cases of the general development (12) of the general term (1). Hence, substituting and integrating

$$\begin{aligned} u = u + \epsilon^{-u}\Sigma \Big\{ & \frac{i_{s_0}}{\omega_{s_0}} R_{s_0} \cos \omega_{s_0} + \ldots . \\ & + \frac{i_{s_0}}{\omega_{s_0} + \Sigma_1^k q\omega_s} \cdot \frac{I}{k} \cdot \Big\{ \tfrac{1}{2}\Sigma_m R_s^m [\Sigma_u (I_{s_0} j_s^m - q i_{s_0} J_s^m) \frac{I}{\omega_s}] \Big\}^k R_{s_0} \\ & \cos (\omega_{s_0} + \Sigma_1^k q\omega_s) + \ldots . \Big\} \\ U = U - \epsilon^{-u}\Sigma \Big\{ & \frac{I_s}{\omega_{s_0}} R_{s_0} \sin \omega_{s_0} + \ldots . \\ & + \frac{I_{s_0}}{\omega_{s_0} + \Sigma_1^k q\omega_s} \cdot \frac{I}{k} \cdot \Big\{ \tfrac{1}{2}\Sigma_u R_s^m [\Sigma_u (I_{s_0} j_s^u - q i_{s_0} J_s^u) \frac{I}{\omega_s}] \Big\} R_{s_0} \\ & \sin (\omega_{s_0} + \Sigma_1^k q\omega_s) + \ldots . \Big\} \end{aligned} \tag{22}$$

These expressions form the complete value of the perturbed or instantaneous elements, and are rigidly correct to any determined order of accuracy. To form the coefficient of the kth term it is only necessary to give the arguments

$$\omega_{s_0} \quad \omega_{s_1} \quad \omega_{s_2} \quad . \quad . \quad . \quad . \quad . \quad \omega_{s_k}$$

every value which exists in the disturbing function R. To form the complete value of the perturbations with any given argument ω_p, it is only necessary to give to $s_0\, s_1 \ldots . s_k$ every value consistent with the condition

$$\omega_{s_0} + q'\omega_{s_1} + q''\omega_{s_2} + \quad . \quad . \quad . \quad . \quad + q^k\omega_{s_k} = \omega_p$$

There exists no difficulty in determining these conditions, and thus calculating the perturbation of any given form to any desired accuracy.

It remains to apply these results to the investigation of the perturbations of Class III., or those due to the indirect effect of the disturbing action of a planet. The problem may be stated thus:—Suppose the disturbing action of a planet to add an additional term to the perturbations of the elements of the lunar orbit, and suppose this additional term to have so small a coefficient that its square may be neglected from its exceeding smallness. Then the existence of this new term will alter the normal disturbing action of the Sun, and in this manner give rise to a series of additional perturbations. It is required to determine them.

Let this additional term which is supposed added to each of the disturbed elements be denoted by

$$\begin{aligned} \delta_1 u &= \mathrm{a}_u \cos\zeta \\ \delta_1 U &= \mathrm{A}_u \sin\zeta \end{aligned} \tag{23}$$

Suppose the preceding additional quantities introduced into the expression for the general term of R; as the square and product of these variations are to be neglected, the additional terms which will arise may be written in the form

$$\begin{aligned} \delta_1 \mathrm{R} &= \Sigma \tfrac{1}{2}\{\Sigma_u (I\,\mathrm{a} + q\,i\,\mathrm{A})\}\mathrm{R}\cos(\zeta + q\mathrm{w}) \\ &= \Sigma\,\phi(\mathrm{A})\,.\,\mathrm{R}\cos(\zeta + q\mathrm{w}) \end{aligned} \tag{24}$$

where, as the function denoted by the symbol ϕ (A) involves only perturbations, it may be regarded as constant in developing R and w.

Substitute in the differential equations, they become

$$\begin{aligned} \frac{\mathrm{d}\,.\,\delta u}{dt} &= -\epsilon^{-u}\Sigma\,.\,i\,.\,\phi(\mathrm{A})\,.\,\mathrm{R}\sin(\zeta + q\mathrm{w}) \\ \frac{\mathrm{d}\,.\,\delta U}{dt} &= -\epsilon^{-u}\Sigma\,.\,I\,.\,\phi(\mathrm{A})\,.\,\mathrm{R}\cos(\zeta + q\mathrm{w}) \end{aligned} \tag{25}$$

These are particular cases of the general form (21), and can be

developed and integrated in exactly the same manner, yielding perfectly analogous expressions for the additional perturbations produced in the values of the elements u and U. The general term for these additional perturbations may be written

$$\delta u = +\epsilon^{-u}[\phi(\mathrm{A}) - \mathrm{a}_u] \cdot \frac{\mathrm{I}}{k} \cdot \left\{ \tfrac{1}{2} \Sigma_m R_s^m [\Sigma_u (I_{s_0} J_s^m - q q^k i_{s_0} J_s^m)] \frac{\mathrm{I}}{\omega_s} \right\}^k \cdot$$

$$\frac{q\, i_{s_0} R_{s_0}}{\zeta + q\omega_{s_0} + \Sigma_1^k q\omega_s} \cos(\zeta + q\omega_{s_0} + \Sigma_1^k q\omega_s) \qquad (26)$$

$$\delta U = -\epsilon^{-u}[\phi(\mathrm{A}) - \mathrm{a}_u] \cdot \frac{\mathrm{I}}{k} \cdot \left\{ \tfrac{1}{2} \Sigma_m R_s^m [\Sigma_u (I_{s_0} J_s^m - q q^k i_{s_0} J_s^m)] \frac{\mathrm{I}}{\omega_s} \right\}^k \cdot$$

$$\frac{I_{s_0} R_{s_0}}{\zeta + q\omega_{s_0} + \Sigma_1^k q\omega_s} \sin(\zeta + q\omega_{s_0} + \Sigma_1^k q\omega_s)$$

This is the second form in which I have calculated the values of these terms of long period due to the disturbing action of the planet.

There is now to be considered the form under which the preceding methods lead to the values for a term of long period due to the disturbing action of a planet.

Returning to the first method, let the direct disturbing action of a planet give rise to a term of long period of the form

$$\mathrm{B} \sin \zeta$$

It is known that the value of the ordinary disturbing function involves every argument which exists in the ordinary perturbations of the Moon arising from the disturbing action of the Sun. Hence the series of arguments in the expression (19) denoted by $(\beta_0 + q\alpha_0)$, representing the arguments comprised in the complete value of the ordinary disturbing function, is identical with the series of arguments denoted by α_0, which represent the corresponding perturbations in the lunar co-ordinates. If, then, the argument of long period be

$$\zeta = \zeta_1 + \zeta_0$$

where by ζ_1 is denoted the portion of the complete argument more directly introduced through the co-ordinates of the planet, and by ζ_0 is denoted the portion more directly introduced through the co-ordinates of the Moon, it is evident, from the similarity in form of R_1 and R_0, that the portion ζ_0 must be identical with one of the arguments comprised in the series β_0 in (17), or with one of the arguments of the ordinary perturbations comprised in the series α_0. As the series of arguments α_1 comprises every argument in the entire series of perturbations produced by the planet, it will comprise every argument in the series $(\beta_1 + q\alpha_0)$; and no others, as the series denoted by $(\beta_0 + q\alpha_0)$ is identical with that denoted by α_0. Hence every term in the perturbations due to the action of the planet must appear in the series $(\beta_1 + q\alpha_0)$.

Consequently, the argument ζ must be one of the arguments comprised in the second term of (19) in the form $(\beta_1 + q\alpha_0)$.

The second term of (19)

$$\Sigma R_1 A_0^q \cos(\beta_1 + q\alpha_0) \tag{27}$$

represents the disturbing function resulting from substituting in the value of R_1 the complete value of the co-ordinates of the Moon as perturbed by the disturbing action of the Sun. Hence, if

$$(R_1 A_0^q)_\zeta \cdot \cos \zeta \tag{28}$$

denote the complete portion of this expression depending on the argument of long period ζ, it will give the complete value of this term as explicitly depending on the direct disturbing function R_1.

There are two methods by which from this term there can be derived the value of the term forming the perturbations with this argument in the Moon's longitude, which has been denoted by B sin ζ, or in other words, by which B can be evaluated.

The first is by substituting this and certain other associated terms in the ordinary differential equations given in my memoir on a "General Method of Treating the Lunar Theory" (*Memoirs*, vol. xliv. 1878), when the final equation to B will be obtained in the form

$$\frac{d}{dt} \cdot \mathrm{B} \sin \zeta = \{ \phi(R_1 A_0) \cdot \} \cos \zeta \tag{29}$$

where the coefficient on the right-hand side is a function depending mainly on the coefficient of the terms with the arguments ζ and $\zeta \pm \alpha$ in the value of the complete disturbing function. Hence ζ becomes the integrating factor, and from the equation

$$\zeta \mathrm{B} = \phi(R_1 A_0) \tag{30}$$

there is derived the value

$$\mathrm{B} = \frac{1}{\zeta} \cdot \phi(R_1 A_0) \tag{31}$$

The other method is equivalent to that adopted by Delaunay in his investigation of the terms of long period arising from *Venus* (*Conn. des Temps*, App., 1862 and 1863). He substitutes the preceding value for the term in the disturbing function with the argument ζ in the differential equation

$$\frac{d\,l}{dt} = -\frac{2}{an} \cdot \frac{d\mathrm{R}}{da} + \frac{e}{2a^2 n} \cdot \frac{d\mathrm{R}}{de} \tag{32}$$

and integrates (*Conn. des Temps*, 1863. Add. p. 41).

In this manner

$$\frac{d}{dt} \cdot \mathrm{B}' \sin \zeta = \phi'(R_1 A_0^q)_\zeta \cos \zeta \tag{33}$$

where the coefficient on the right-hand side is a simple function of the coefficient of the argument ζ in the disturbing function as developed above. Hence, as before, the integrating factor is ζ, and

$$\zeta B' = \phi'(R_1 A_o^q)_\zeta \tag{34}$$

or

$$B' = \frac{1}{\zeta}\phi'(R_1 A_o^q)_\zeta \tag{35}$$

The coefficient B′ thus determined is not exactly the same as B previously found; as B′ is the coefficient of the term of this form in the mean longitude, whilst B is the coefficient of the term in the true longitude. The two, however, only differ by small quantities, unimportant for the present purpose.

Subsequently it will be shown that this last result, or the value B′, is the result which Delaunay obtained for the value of the coefficient of the terms of long period arising from the disturbing action of *Venus*.

In reality, however, neither of these equations, (30) or (34), is complete.

This can be shown as follows.

When the values of B or B′ resulting from the previous equations (30) and (34), and forming what is really only a first approximation to the complete value, come to be substituted in the complete disturbing function, they give rise to the terms included in the third portion of the disturbing function as developed in (19). This portion may be written somewhat more in detail under the form

$$\begin{aligned}&\Sigma\{R_o[a_1 + q_1 b_1]\cos(\beta_o + q_1\alpha_1)\\ &+ R_o[a_1 + q_1 b_1][a_o + q b_o]\cos(\beta_o + q\alpha_o + q_1\alpha_1)\\ &+ \tfrac{1}{2}R_o[a_1 + q_1 b_1][a_o + q b_o][a_o' + q' b_o']\cos(\beta_o + q\alpha_o + q'\alpha_o' + q_1\alpha_1)\\ &+ \ldots\ldots\end{aligned} \tag{36}$$

Now the series of coefficients denoted by b_1, being the series of coefficients of the perturbations in longitude due to the disturbing action of the planet, they will contain the coefficient B, and thus give rise to the series of terms

$$\begin{aligned}\Sigma . R_o[q_1 B]\{&\cos(\beta_o + q_1\zeta)\\ &+ [a_o + q b_o]\cos(\beta_o + q\alpha_o + q_1\zeta)\\ &+ [a_o + q b_o][a_o' + q' b_o']\cos(\beta_o + q\alpha_o + q'\alpha_o' + q_1\zeta)\\ &+ \ldots\ldots\end{aligned} \tag{37}$$

Then this series of terms will not disappear from the value of either

$$\frac{dR}{da}$$

or

$$\frac{dR}{dl}$$

as the argument β_0 will involve l, and R_0 will involve a, and in forming these differentials a_0 b_0 a_0 are to be regarded as constants.

Now β_0 and α_0 represent the same series of arguments, so for each argument in the series β_0 there will be an identically similar argument in the series α_0. Hence one of the pairs of arguments denoted by

$$\beta_0 + q\alpha_0$$

will vanish in each case, and reduce the second term of (37) to the form

$$R_0[q_1 B]\{[a_0 + qb_0]\}\cos(q_1\zeta) \tag{38}$$

As a_0 and b_0 will not be equal, it follows that this coefficient cannot vanish whichever sign be given to q.

As R_0 a_0 b_0 are all functions of one or more of the small quantities commonly denoted by m, e, ... η, and of the second order at least, the above term may be written in the form

$$\{\chi_1(m \ldots .)^4 + \chi'_1(m \ldots .)^5 + \ldots .\}[B]\cos\zeta. \tag{39}$$

Similarly, there will be a number of combinations of the arguments of the series β_0 α_0 α_0' which will make

$$\beta_0 + q\alpha_0 + q'\alpha'_0 = 0$$

and thus reduce the third term of the preceding value for this portion of the disturbing function to the form

$$R_0[q_1 B]\{\tfrac{1}{2}[a_0 + qb_0][a'_0 + q'b'_0]\}\cos(q_1\zeta) \tag{40}$$

or, as it may be written,

$$\{\chi''_1(m \ldots .)^6 + \ldots .\}[B]\cos\zeta. \tag{41}$$

The existence of these terms renders both of the equations (30) and (34) previously given for determining B and B′ incomplete. The first of them (30) should be written in the form

$$\zeta B = \phi(R_1 A_0) + \{\psi(m \ldots .)^4 + \psi_1(m \ldots .)^5 + \ldots .\}B$$

and the integrating factor instead of ζ is

$$\zeta - \{\psi(m \ldots .)^4 + \psi_1(m \ldots .)^5 + \ldots .\} \tag{42}$$

or

$$B = \frac{\phi(R_1 A_0)}{\zeta - \{\psi(m \ldots .)^4 + \psi_1(m \ldots .)^5 + \ldots .\}} \tag{43}$$

and the second of them (34) should be

$$\zeta B' = \phi'(R_1 A_0^q)_t + \{\psi'(m \ldots .)^4 + \psi'_1(m \ldots .)^5 + \ldots .\}B \tag{44}$$

where the integrating factor instead of ζ is

$$\zeta - \{\Psi(m \ldots .)^4 + \Psi_1'(m^5 \ldots .)^5 + \ldots .\} \tag{45}$$

or

$$B' = \frac{\phi'(R_1 A_0 q)_\zeta}{\zeta - \{\Psi(m \ldots .)^4 + \Psi_1'(m \ldots .)^5 + \ldots .\}} \tag{46}$$

These equations (43) and (46) represent the values found by myself, and, I have no doubt, by Hansen.

The second method of investigating these terms, or the method of perturbed elements, founded on equation (20) as developed in expressions (22) and (26), leads to exactly the same result. For still further brevity put

$$\Sigma_u (I_{s_0} j_{s_i}^m - q q' i_{s_0} J_{s_i}^m) \frac{1}{\omega_{s_1}} = q'[s_0 s_i]^m \tag{47}$$

then suppose (22) to represent the development of R, the disturbing function due to the direct action of the planet, it may be written in the form

$$\begin{aligned} u = u + \epsilon^{-u} \cdot \Sigma \cdot [R_1]_{s_0} \cdot \Big\{ & \frac{i_{s_0}}{w_{s_0}} \cos \omega_{s_0} + \ldots .. + \frac{i_{s_0}}{\omega_{s_0} + \Sigma_1^k q \omega_s} \\ & \cdot \frac{1}{\lfloor \underline{\kappa}} \{q'[s_0 s]^m\}^k \cos (\omega_{s_0} + \Sigma_1^k q \omega_s) + \ldots .. \Big\} \\ U = U - \epsilon^{-u} \Sigma \cdot [R_1]_{s_0} \Big\{ & \frac{I_{s_0}}{\omega_{s_0}} \sin \omega_{s_0} + \ldots .. + \frac{I_{s_0}}{\omega_{s_0} + \Sigma_1^k q \omega_{s_0}} \\ & \cdot \frac{1}{\lfloor \underline{\kappa}} \{q'[s_0 s]^m\}^k \sin (\omega_{s_0} + \Sigma_1^k q \omega_s) + \ldots .. \Big\} \end{aligned} \tag{48}$$

Then if ζ be the argument of a term of long period due to the disturbing action of the planet, these expressions will lead to the following values for the perturbations of the element:

$$\begin{aligned} a_u \cos \zeta &= + \frac{1}{\zeta} \cdot \epsilon^{-u} \cdot \Big\{ i[R_1] + \ldots .. + i_{s_0} \cdot \frac{1}{\lfloor \underline{\kappa}} \rightsquigarrow t_{q^m} \Big\} + \ldots .. \Big\} \cos \zeta \\ A_u \sin \zeta &= - \frac{1}{\zeta} \cdot \epsilon^{-u} \cdot \Big\{ I[R_1] + \ldots .. + I_{s_0} \cdot \frac{1}{\lfloor \underline{\kappa}} \{q'[s_0 s]^m\}^k + \ldots .. \Big\} n \end{aligned} \tag{49}$$

with the condition that $s_0\ s_1 \ldots .. s_k$ are to be given every value consistent with the condition that

$$[\omega_1]_{s_0} + q' \omega_{s_1} + q'' \omega_{s_2} + \ldots .. + q^k \omega_{s_k} = \zeta$$

The argument ω_{s_0} has been written $[\omega_1]_{s_0}$ to show that it alone represents an argument derived from the series forming R, the remaining argument being derived from the expression for the ordinary perturbations produced by the Sun.

These values of the perturbations correspond to the values

(30) and (34), and represent the direct action of the planet. They are equivalent to those given by Delaunay.

But it has already been shown that when these values come to be substituted in the disturbing function R_o representing the direct action of the Sun, they give rise to the series of additional terms (26). The general term of these additional perturbations can be written in the form

$$\delta u = +\epsilon^{-u}[\phi(A)-a_u]\cdot\frac{1}{\underline{|k}}\cdot\{q'[s_o s]^m\}^k\frac{q\,i_{s_o}R_{s_o}}{\zeta+q\omega_{s_o}+\Sigma_1^k q\omega_s}\cos(\zeta+q\omega_{s_o}+\Sigma_1^k q\omega_s)$$
$$\delta U = -\epsilon^{-u}[\phi(A)-a_u]\cdot\frac{1}{\underline{|k}}\cdot\{q'[s_o s]^m\}^k\frac{I_{s_o}R_{s_o}}{\zeta+q\omega_{s_o}+\Sigma_1^k q\omega_s}\sin(\zeta+q\omega_{s_o}+\Sigma_1^k q\omega_s) \quad (50)$$

But the series of arguments by $\omega_s \ldots\ldots\ldots \omega_{sk}$ are identical, so that for a considerable number of combinations in the series there will hold the condition that

$$q\omega_{s_o}+q'\omega_{s_1}=0$$
$$* \qquad *$$
$$q\omega_{s_o}+\Sigma_1^k q\omega_s=0$$

whilst i_{s_o} and I_{s_o} do not vanish. Hence the above expressions will reduce to

$$a_u\cos\zeta = +\epsilon^{-u}[\phi(A)-a_u]\cdot\frac{1}{\underline{|k}}\{q'[s_o s]^m\}k.\ \frac{q\,i_{s_o}R_{s_o}}{\zeta}\cdot\cos\zeta$$
$$A_u\sin\zeta = -\epsilon^{-u}[\phi(A)-a_u]\cdot\frac{1}{\underline{|k}}\{q'[s_o s]^m\}k.\ \frac{I_{s_o}R_{s_o}}{\zeta}\cdot\sin\zeta \quad (51)$$

From (24), the right-hand side of this expression will involve both a_u and A_u, and R and $[s_o\, s]$ are each of the order $[m\ldots]^2$ at least, so that the preceding expression may be written in the form

$$\zeta a_u = \{\Psi[m\ldots.]^4+\Psi_1[m\ldots.]^5+\ldots.\}[a_u+f(a.A.)]$$
$$\zeta A_u = \{\Psi'[m\ldots.]^4+\Psi'_1[m\ldots.]^5+\ldots.\}[A_u+f(a.A)] \quad (52)$$

and the form of the integrating factors be

$$\zeta - \{\Psi[m\ldots.]^4+\Psi_1[m\ldots.]^5+\ldots.\} \quad (53)$$

the same as already found in (42) and (45).

The simplicity of the differential equations when the method of perturbed elements is employed in the preceding manner enables a further step to be made. By developing the function denoted by

$$\{q'[s_o s]^m\}^k$$

it is possible to show that the terms involving the first of this series of functions will vanish from the preceding expression for the additional perturbations (51), as $q'[s_o s_1]$ will be zero. Hence

there can be no term of the preceding kind in the longitude of the Moon which depends on the square of the disturbing force of the Sun, so that we have

$$\Psi\,[m\ldots\ldots]^4 = 0$$
$$\Psi_1\,[m\ldots\ldots]^3 = 0$$

This is an extension of a theorem already established with respect to other terms by Poisson and Pontécoulant.

But the same does not hold good with the terms depending on the cube of the disturbing force of the Sun, or involving positive powers of ζ, for the functions

$$q'q''\,[s_0\,s_1]\,[s_0\,s_2]$$

or the function

$$q'\,[s_0\,s_1]''$$

depending on the perturbation of the order of the square of the disturbing force, they do not vanish. Hence the true form of the integrating factors (53) (42) and (45) take the form

$$\begin{aligned}
&\zeta - \{\Psi_2\,[m\ldots\ldots]^6 + \Psi_3\,[m\ldots\ldots]^7 + \ldots\ldots\}\\
&\zeta - \{\psi_2\,[m^6\ldots\ldots]^6 + \psi_3\,[m\ldots\ldots]^7 + \ldots\ldots\}\\
&\zeta - \{\psi'_2\,[m\ldots\ldots]^6 + \psi'_3\,[m\ldots\ldots]^7 + \ldots\ldots\}
\end{aligned} \qquad (54)$$

In the case of the two last of these, this has been verified by actual computation, the term of the orders m^4 and m^5 vanishing, but those of the orders m^6 and m^7 remaining.

The preceding explanation of the methods I have employed in these investigations and the results I have obtained, does not profess to be more than a mere sketch, given to enable the principles of the method to be clearly understood. No attempt has been made to supply the necessary details of the transmission from one step to another, to dwell on any points of interpretation that are involved, or to write out the expressions in sufficient length to enable more to be seen than the general principle. To have done more would have been to extend the space occupied to at least ten times its present amount; and it is unnecessary, as the whole details are embraced in my nearly completed memoir on the theory of these terms of long period. For the present purpose all that is important is the principles on which the results are founded; the details of the transformations and expansions into proper form must be taken as accurate.

It remains, then, to critically examine Delaunay's method of investigating these terms, and to consider certain objections raised to the conclusions submitted in my Paper on "Hansen's Terms of Long Period" (*Monthly Notices*, vol. xxxviii. pp. 268–279, March 1878).

Objection has been raised to my conclusions in that Paper on the ground that they depend on one of the older methods of investigating this subject, whereas Delaunay's results are based

on his own *far more rigorous method*. In the words of a very distinguished astronomer, "You speak of the three parts into which the action of *Venus* on the disturbed motion may be divided. Now this division is a result of the old and imperfect way of viewing the subject; that is, of considering the analytic expressions which give the position of the Moon relative to the Earth as being made up of two parts, certain elliptic terms and certain terms introduced by the action of the Sun. Now the elegance and beauty of Delaunay's method consists in this: that his final equations which express the three co-ordinates of the Moon in terms of these arbitrary constants are not so divided at all, but are *rigorous expressions* for the motion of the Moon around the Earth under the disturbing action of the Sun. Therefore, when by the general equations of dynamics he determines the variations of these constants produced by the action of the planet, the proceeding is rigorous and complete, and that is the end of it." Further, it is claimed that in Delaunay's method of procedure the "differential equations of motion contain all the terms, and any rigorous analytic process of integrating these equations must lead to rigorous results. Now Delaunay *includes all terms of every class in every step*. Hence his results must be correct, unless he has omitted some term through inadvertence."

Now I object to the very basis of these criticisms. I deny absolutely that Delaunay's method of transformed elements is in any manner more rigorous than the methods of perturbed co-ordinates or perturbed elements that have been employed by others. It may be more elegant, it may be more beautiful, it is certainly not more rigorous, whilst it is equally certain it is less simple and less direct than the method of perturbed elements depending on the direct integration of the system of equations (20).

As actually employed in the calculation of the perturbations, or even as capable of so being employed, there is not one of these methods that is strictly rigorous, for all involve numerous approximations, and to the same order of approximations all are equally rigorous and all yield identically the same result.

Though the fundamental equations and transformations involved in Delaunay's method are, I admit, sufficiently rigorous, this is no more than is true to the same degree of the less direct equations and transformations of the older methods; and when Delaunay's method comes to be actually applied, like these others, it sinks into a method of successive approximations where whole series of terms have to be neglected, on the common ground that they appear too small to be of importance. It is true that it is claimed that Delaunay's final equations—that is, the equations after, by successive transformations, all the periodical terms have been removed from the disturbing functions—admit of *rigorous* solution. They admit of *finite* solution, certainly, but scarcely of rigorous solution, as this first stage can only be reached by neglecting whole masses of terms, on

the sole ground that they are beyond a certain arbitrarily fixed limit of supposed smallness. But, exactly the same holds true of the older methods, where, if only you neglect all the terms beyond this same arbitrarily fixed limit, there can be obtained a similarly finite solution.

In truth, the only real difference in principle between Delaunay's method and the method of perturbed elements employed by myself consists in the fact that, whereas I commence by explicitly developing in an algebraical form the entire series of distinct operations, and then replace the algebraical symbols denoting the perturbations as they are successively determined in large masses at a time; on the other hand, Delaunay does not obtain any such algebraical development of his proposed operations, but contents himself with obtaining a new expression for the disturbing function, as he calculates separately the value of the perturbations produced by each distinct term in the expression he successively obtains for this disturbing function. Both, properly conducted, must be equally rigorous. Indeed, if instead of employing the actual values in terms of $(a.e.\eta.n.....)$, Delaunay's method could be employed with algebraical coefficients, there would be derived the same expression for the disturbing function as is obtained as my first step in the method of perturbed elements.

It is scarcely necessary, however, to further discuss this supposed superiority of Delaunay's method, because, whatever preference may be given to it by certain astronomers, it must be admitted that any of the older methods employed by Poisson, Pontécoulant, and Hansen, properly applied, is capable of yielding quite as accurate a solution as Delaunay's, so that if these methods unite in giving a distinct series of terms in addition to those obtained by Delaunay, it is certain that Delaunay has not properly applied his method to this portion of his investigations. Admittedly, the error does not lie in the principle of the method, but in the manner of its application.

It remains to examine the exact manner in which Delaunay has calculated the value of these terms of long period due to the action of the planet *Venus*.

It is generally supposed, as is indicated by the quotations already given, that Delaunay has calculated these terms by a strict application of the method of transformed elements employed by him for determining the ordinary lunar perturbations due to the direct disturbing action of the Sun. But a critical examination of his investigations, published in the *Connaissance des Temps*, Additions 1862 et 1863, will show that this is not the case. For present examination the more important investigation may be selected in the volume for the year 1863.

Delaunay first obtains an expression for the ordinary disturbing function expressing the action of a planet upon the Moon, and corresponding to that designated R, though he retains terms depending on the inclination of the planet to the ecliptic.

These he proceeds to develop in terms of r, $r_\circ$ ρ. $V_\circ$ V, and the disturbed co-ordinates of the Moon, V and r. He then proposes to replace r^2 and V by their proper values in the actual orbit of the Moon as disturbed by the Sun. This is nominally done by the application of the system of transformation of elements represented by the 492 operations and supplementary operations of his development of the ordinary lunar theory. Suppose this properly done, and the constants replaced by their final values, what would be obtained is purely the second term of the expression (19), or exactly what would have been obtained had he directly replaced r. V..... by their values as the co-ordinates of the Moon as disturbed by the Sun. The expression which forms his final value for R, the disturbing function expressing the action of the planet, should be identically the same expression as I have written in the form

$$\Sigma R_1 A^q \cos(\beta_1 + q\alpha_\circ). \tag{55}$$

If the same final constants be adopted these two expressions should be identical, term by term, factor by factor.

Delaunay does not retain, however, all the terms of this expression, or, indeed, even attempt to calculate them, but neglects every term except that depending on the particular argument of the term of long period he is calculating. That is, he confines himself to calculating merely the term denoted by me,

$$(R_1 A^q_\circ)_\xi \cos \zeta, \tag{56}$$

or rather the series of allied terms which may be regarded as grouped under the single argument ζ, though differing slightly in the constant portion of their argument.

But really even as much as this is not done. Practically, Delaunay confines himself (*Conn. des Temps*, 1863, App. pp. 7 et 8) to merely writing

$$-a^2 \cdot \tfrac{19}{8} \cdot \frac{n'^2}{n^2} \cdot \frac{a'^3}{r'^3} \cdot \cos(2V') \tag{57}$$

for

$$r^2 \cos(2V),$$

which is simply tantamount to neglecting the entire effect of the perturbations of the Moon save those depending on the very first term of the *Variation*.

Delaunay then proceeds to replace

$$\rho r_1 r_\circ V_1 V_\circ$$

by their elliptic values, so as to obtain in detailed form the twelve distinct terms which have been represented, hitherto, by the single term

$$R_1 \cos \zeta,$$

but which really are of the form

$$R_n \cos\{\zeta + (\lambda)_n\}, \tag{58}$$

where $(\lambda)_n$ is a function of the constants denoting the longitude of the perihelia and nodes of the Earth and *Venus*, and differs for each of the twelve terms, although ζ the portion which varies with the time may be regarded as identical in all.

Thus the expression for the values of these terms in the disturbing function which is obtained by Delaunay in this manner is identical with that which can be obtained from (55), the second term of (19), by neglecting all save the first power of the disturbing force of the Sun, and every term in the perturbations of even this order save the factors of the lowest order in the *Variation*. Strictly, it in no sense depends on his method of transposed elements, though made to appear as if it does; nor does it differ in any way from the values hitherto employed by astronomers save in being somewhat less complete.

Delaunay now proceeds to substitute this value of these terms in the disturbing function, in the differential equations adopted by him for determining the value of the resulting perturbations. But he does not employ, as generally supposed, and assumed in the criticisms of my Paper already quoted (page 422), the method of transformed elements which he has used in the other parts of his lunar theory, and founded on the system of equations given on pages 65 and 84, 85 of his 'Théorie de la Lune' (*Mémoires de l'Académie des Sciences*, tom. xxxiii.); but he proceeds to adopt the ordinary differential equations for the variation of elements where the disturbing function R has its ordinary meaning, and does not denote that special value after it has undergone the system of transformation of elements necessary to remove the portion explicitly depending on the ordinary lunar perturbations.

In fact, Delaunay determines the required perturbations from the ordinary form for the differential equation to the variation of the mean longitude l, or from (*Conn. des Temps*, App. 1863, pp. 41 et 42)

$$\frac{d\,l}{d\,t} = -\frac{2}{a\,n}\cdot\frac{d\,R}{d\,a} + \frac{e}{2\,a^2\,n}\cdot\frac{d\,R}{d\,e}, \tag{59}$$

or practically, as he neglects the second term, from the equation

$$\frac{d\,l}{d\,t} = -\frac{2}{a\,n}\cdot\frac{d\,R}{d\,a}, \tag{60}$$

where he regards a, after differentiating R with respect to a, as being a constant, making the assumption that, as the argument of the term of long period is explicitly independent of l, the mean longitude of the Moon, and as a the mean distance is to be determined from the differential equation

$$\frac{d\,a}{d\,t} = \frac{2}{a\,n}\cdot\frac{d\,R}{d\,l}, \tag{61}$$

there can exist no term with this argument in the value of a, so that this element may be regarded as constant in the preceding equation (60).

Then, writing for convenience the terms in the disturbing function which have this argument in an analogous form to Delaunay, or in the form

$$a^2 \Sigma(B' + B'' m^2) A_n \cos \{\zeta + (\lambda)_n\}, \tag{62}$$

Delaunay, by substitution in the equation (60), obtains expression corresponding to the forms

$$\frac{d l}{d t} = -\frac{1}{n} \Sigma(4 B' + 10 B'' m^2) A_n \cos \{\zeta + (\lambda)_n\} \tag{63}$$

and

$$\delta l = -\frac{1}{\zeta} \Sigma(4 B' + 10 B'' m^2) A_n \sin \{\zeta + (\lambda)_n\} \tag{64}$$

(*Conn. des Temps*, App. 1863, p. 42, formule (*s*)).

This is the value which Delaunay finds for the portion of the coefficient of this term depending on the direct disturbing action of the planet. The portion depending on the more indirect action of the planet as afforded by the perturbations in the mean distance and eccentricity of the Earth's orbit, is then calculated in a similar manner by means of the same differential equations. By uniting these two portions, Delaunay obtains the final value for the term.

The preceding account shows clearly that after calculating the value of R, or rather of these terms in R, in a form corresponding exactly with the value assigned by any of the ordinary methods of developing the disturbing function, Delaunay altogether abandons his special method of transformation of elements, and proceeds to integrate his equations after the ordinary method of variation of elements adopted by Ponté-coulant and Hansen.

Out of the numerous assumptions involved in the preceding method by which Delaunay has determined the value of the perturbations with this argument of long period in the expression for the mean longtitude, from the values (58) found by him for the corresponding terms in the disturbing functions, there are two which deserve particular attention. They are :

First, the assumption made in the integration of equation (60), that as the argument ζ is independent of l, the mean longitude of the Moon, it follows that neither

$$\frac{d \mathrm{R}}{d l}$$

nor

$$\frac{2}{a n} \cdot \frac{d \mathrm{R}}{d l}$$

can, either of them, contain a term with the argument ζ; so that the perturbations δa of the mean distance must be free from any such term.

Second, the assumption implicitly involved in his entire treatment of the differential equation to the mean longitude (59), namely that the value of the disturbing function R he has determined in (58) is the complete value of R involved in the differential equation (59), as it is integrated by him so as to obtain the value (64).

The first of these assumptions is certainly unjustifiable, except for terms depending on the first power of the disturbing force. For even if it were admitted that the function

$$\frac{d\,\mathrm{R}}{d\,l}$$

could not contain such terms, it would contain terms of the second order with respect to m and with the argument

$$2\,\theta \pm \zeta,$$

where $2\,\theta$ denotes the argument of the *Variation.* Hence the expression for δa must also contain terms of the order m^2 with the same argument. Consequently there will arise in the equation

$$\frac{d\,a}{d\,t} = \frac{2}{a\,n} \cdot \frac{d\,\mathrm{R}}{d\,l}$$

terms of the form

$$\frac{d\,a}{d\,t} = \phi(m^4)\,\mathrm{B}\sin\zeta \tag{65}$$

and the perturbations of the mean distance will contain terms of the form

$$\delta a = -\frac{1}{\zeta} \cdot \phi(m^4)\,\mathrm{B}\cos\zeta. \tag{66}$$

Now as ζ, the integrating factor of this term of long period, may be regarded as likewise a quantity of the fourth order, it follows that the term in (60) arising from the perturbations in δa will be of the same importance as those arising directly from the function

$$\frac{d\,\mathrm{R}}{d\,a}.$$

Delaunay has neglected these terms without any attempt to show they are insensible. Hence his result is imperfect.

But the assumption that terms of this form cannot exist in the function denoted by

$$\frac{d\,\mathrm{R}}{d\,l} \tag{67}$$

is founded on the supposition that

$$\frac{d\,R}{d\,l} + \delta\frac{d\,R}{d\,l} = \frac{d\,R}{d\,l} + \frac{d\,\delta R}{d\,l} \tag{68}$$

which is not true, as the meaning to be assigned to l on the two sides is entirely different, one being the undisturbed or elliptic value and the other the disturbed value. It can easily be shown that terms with the argument ζ will arise in (67) in exactly the same manner as they arose in (65).

It is true that it might be shown that certain of these terms are mutually destructive, but Delaunay has made no attempt to establish this, and it is not true of all these terms, nor completely true of any of them.

It remains to consider the second and more serious assumption made by Delaunay, as it is really the fundamental cause of the difference between the results obtained by him and those obtained by other astronomers using independent methods.

Returning to the expansion of the disturbing function in terms of the elliptic elements and their perturbations, or of the elliptic co-ordinates and their perturbations (19); merely remarking *en passant*, that in these investigations Delaunay uses exactly the same division between the two as employed by myself, though the fact seems to have escaped the attention of the distinguished author of the criticism I have quoted on page 422; I have shown that the complete value of the disturbing function may be conveniently divided into three portions. That is, we may write

$$R = R' + R'' + R''' \qquad (69)$$

where these three portions represent the three lines of the expression (19) for R, and

- R' denotes the portion expressing the sole action of the Sun.
- R'' denotes the portion expressing the action of the planet on the motion of the Moon as disturbed by the Sun.
- R''' denotes the portions expressing the action of the Sun on the additional perturbations of the Moon due to the action of the planet.

Now it has been shown that both R'' and R''' contain terms depending on the argument of long period ζ, and so must be both retained in determining the values of the perturbations of this form, though, as R' does not contain such terms, its consideration may be neglected. Hence the value of the disturbing function which must be employed is

$$R = R'' + R'''$$

Next, the ordinary equations for determining the perturbations by the method of the variation of elliptic elements, by considering only the lowest terms, may be written in the following form:—

$$\begin{aligned} \frac{d\delta a}{dt} &= +\frac{2}{an} \cdot \frac{dR}{dl} \\ \frac{d\delta l}{dt} &= -\frac{2}{an} \cdot \frac{dR}{da} + \frac{e}{2a^2 n} \cdot \frac{dR}{de} \end{aligned} \qquad (70)$$

writing down only the first of the three pairs of terms forming the entire set of equations. Hence, on replacing the disturbing function R by its complete value in so far as it involves these terms depending on the perturbing action of the planet, that is by

$$R = R'' + R''' \tag{71}$$

where Delaunay's factor

$$\frac{\mu}{2a}$$

has been regarded as incorporated in R'', so as to bring the expressions into harmony with those of Delaunay, the resulting system of differential equations may be written as

$$\begin{aligned} \frac{d\delta a}{dt} &= \left\{ +\frac{2}{an} \cdot \frac{dR''}{dl} \right\} + \left\{ +\frac{2}{an} \cdot \frac{dR'''}{dl} \right\} \\ \frac{d\delta l}{dt} &= \left\{ -\frac{2}{an} \cdot \frac{dR''}{da} + \frac{e}{2a^2 n} \cdot \frac{dR''}{de} \right\} + \left\{ -\frac{2}{an} \cdot \frac{dR'''}{da} + \frac{e}{2a^2 n} \frac{dR'''}{de} \right\} \end{aligned} \tag{72}$$

These are the forms which must be used to obtain the complete values of the perturbations of this kind. On integration they lead at once to the values (43) and (46), which have been already given as representing the complete values of these terms as found by Hansen and myself.

Now it has been shown that Delaunay employs these identical equations (70), and obtains his results by integrating them, not according to his own special method, but by integrating them in the ordinary method, such as would have been employed by Poisson or Leverrier. But, when so used and so integrated, it is essential to employ the complete value of the disturbing function given by (71). Delaunay does not do this, for, as it has already been shown, he uses simply the value of the disturbing function which has been denoted by R''. Consequently, the results he obtains are identical with that given by merely the first half of the equations (72), which are exactly the same as the incomplete values (31) and (35).

Hence, Delaunay has neglected the entire series of terms denoted by R''', and the results he has obtained are incomplete. In other words, Delaunay has completely neglected the very class of terms depending on the first power of the disturbing force of the planet, and the square and cube of the disturbing force of the Sun, from which Hansen derives the large values of this class of terms of long period.

This conclusion forms the complete justification of my contention on page 407, and serves to establish the accuracy of the results arrived at in my Paper on "Hansen's Terms of Long Period," in the *Monthly Notices* for March 1878.

Throughout the preceding discussion on pages 425–429 it has been assumed that the equations ultimately employed by Delaunay

in his "Memoir" are meant to be the ordinary differential equations for the variation of elliptic elements, such as are made use of for similar purposes by Poisson (*Mémoires de l'Académie*, tome xiii.), and by Pontécoulant (*Système du Monde*, tome iv.), and by Delaunay himself, for calculating the secular acceleration in the mean motion of the Moon (*Conn. des Temps*, Add. 1861).

Such I believe to have been Delaunay's intention, though it is true that in his investigation he refers solely to his only published Paper on the *Additions de la Connaissance des Temps pour* 1861, pages 7 et 9, a paper on his own special method; a reference the more puzzling as the equations which he uses, and to which he refers, do not appear in this Paper in the form in which he employs them.

To me, Delaunay's idea seems to have been, that having succeeded by his series of transformations in getting rid of all the periodical terms in the portion of the disturbing function depending solely on the Sun, denoted here by R_0, he could regard the differential equations as reduced to expressions depending solely on his final constants (a e γ), and the differentials of the function denoted here by R_1. Hence by neglecting the powers and products of the quantities (e γ n), he was able to regard his equations as reduced to the ordinary differential equations of the method of the variation of elliptic elements, and integrate them accordingly.

But it may be urged that this is not really the case, but that Delaunay intended the ultimate equations (59) and (61) to be regarded as the final equations of his special method of transformed elements, after by successive operations there had been removed all the periodical terms of the portion R_0 of the complete disturbing function which depends solely on the perturbing action of the Sun.

Delaunay has not given these equations in this form as explicit functions of these constants and the differentials of R_1, but they are given by Prof. G. W. Hill on page 245 of his fine memoir on the *Lunar Inequalities due to the Ellipticity of the Earth* (*Astronomical Papers of the American Nautical Almanac*, vol. iii. part ii.). By neglecting terms depending on powers and products of (e γ n), these differential equations may be reduced to the same form as (59) and (61).

Hence it may be urged that the ultimate equations actually used by Delaunay are really the fundamental equations of his special system, so that in varying the method adopted for integrating them from the method of transformation to the method of direct integration, he has only employed a perfectly legitimate variation.

But I do not think that this contention can be regarded as accurate.

The non-periodical portions forming the coefficients of the differentials of R in these final equations of Delaunay's represent merely the non-periodical portion of the functions of the disturbed

elements ($a\ e\ \gamma$) forming the coefficients of the differentials of R in the ordinary equations for determining the variations of the elliptic elements in the disturbed orbit. Hence these final equations of Delaunay's are merely equivalent to the result obtained by completely developing these coefficients in the ordinary differential equations of this form as employed by Poisson or Pontécoulant, by replacing ($a\ e\ \gamma$) by their disturbed values, and then neglecting all the periodical terms. Thus if, in the coefficients of the differentials of R'' in expression (72), the elements ($a\ e\ \gamma$) be replaced by their complete disturbed values, and the periodical terms be neglected, they will become identical with the corresponding final equations of Delaunay.

Were it permissible, then, to regard the equations (59) and (61) employed by Delaunay in his investigation as merely simplified forms of his final equations, which could be legitimately integrated in the ordinary manner by regarding ($a\ e\ \gamma$) as constants, it would follow that they should lead to results identically equivalent to those given by the first half of the equations (72). In fact, thus integrated, these two expressions do lead to identically the same results, the values actually obtained by Delaunay in his memoir, and corresponding in all respects with the values (31) and (35). But for this to be legitimate, it must follow that the second half of the equations (72) must vanish, and this can only be by the constituent terms mutually destroying each other, and so reducing the whole function to zero.

It is evident, however, that this is impossible.

Moreover, this second part of the equations (72) is the portion which yields the perturbations arising from the disturbing action of the Sun upon the perturbations in the motion of the Moon due to the disturbing action of the planet. Hence, to suppose it to be zero would be equivalent to the theorem that the effect of the disturbing action of the Sun upon the motion of the Moon is independent of any inequalities in this motion due to the action of a planet, or, as these comprise inequalities of every period and all kinds, that it is independent of the existence of any inequalities whatever. This is obviously untrue. Consequently it follows that the second portion of these equations (72) cannot vanish through mutual destruction, and so must not be neglected. Therefore, the quantities ($a\ e\ \gamma$) forming the coefficients of the differentials of R in Delaunay's final equations cannot be regarded as constant, but must be held to be variable, each representing a series of perturbations due to the disturbing action of the planet. But, if variable, the method of integration actually employed by Delaunay is imperfect, and can lead only to imperfect results; he having failed, thus, to take into account the whole class of terms depending on the disturbing action of the Sun and planet, which is represented by the function of R'''.

It is not difficult to see that the only legitimate ways of integrating the equations (59) and (61), if they be regarded as forms of Delaunay's final equations, are either to employ the

method of transformed elements, and remove by a series of operations all the periodical terms from R_1, the disturbing function depending on the planet, or to replace $(a\,e\,\gamma)$ by their complete values as perturbed elements, and integrate the result in the ordinary manner.

However, it is not necessary to consider the exact nature of the results which would be obtained by either of these methods, for it can be seen that neither of them would lead to the same result as given by the complete value of equations (72), unless an entirely distinct form be given to the expressions for the perturbation of the quantities $(a\,e\,\gamma)$ to that contemplated by Delaunay, and which alone can be derived legitimately from Delaunay's expressions by regarding as complete the value to which ultimately he reduces $R_\circ$ by his series of transformations.

Hence there must be some error of principle in the method by which Delaunay thus reduces this portion $R_\circ$ of the complete disturbing function to the form

$$\psi(a\,.\,e\,.\,\gamma)$$

where these quantities are variables, supposed capable of being regarded as constants with respect to the periodical perturbations of solar origin, or to be independent of any periodical terms depending on the disturbing action of the Sun.

It remains to point out the origin of this error, and to show clearly the manner in which it has led Delaunay to omit a whole series of terms which ought to have been retained.

This may be best done, perhaps, in the following way:—

Consider, at first, merely the perturbations depending solely on the disturbing action of the Sun. Then the values of the elements of the Moon's orbit may be written in the form

$$\begin{aligned} \mathrm{u} &= u + \Sigma_1^\infty \delta_\circ u \\ \mathrm{U} &= U + \Sigma_1^\infty \,.\delta_\circ U \end{aligned} \tag{73}$$

where

u U denote the non-periodical parts of the variables u U.

$\Sigma_1^\infty \delta_\circ u$, $\Sigma_1^\infty \delta_\circ U$ denote the series of periodical terms arising solely from the disturbing action of the Sun.

Employing these variables, the disturbing function must be written in the form

$$R_\circ = M_\circ \Sigma_\circ^\infty \psi_\circ(\mathrm{u}\,.\,\mathrm{U}\,\ldots\ldots)_n \tag{74}$$

where the mass of the Sun is denoted by $M_\circ$.

Delaunay's method consists in replacing the variables u U by the new variables u′ U′, such that

$$\begin{aligned} \mathrm{u}' &= \mathrm{u} - \delta_\circ u_1 \\ \mathrm{U}' &= \mathrm{U} - \delta_\circ U_1 \end{aligned} \tag{75}$$

where

$$\delta_\circ u_1 \qquad \delta_\circ U_1$$

are so determined that when substituted in R_o they cause the disappearance of the term with the argument$_1$, so that the new value of the disturbing function takes the form

$$R'_o = M_o \psi'_o(u' \dots)_o + M_o \Sigma_2^\infty \psi'_o(u'U' \dots)_n \tag{76}$$

from the values of the new variables $u'\ U'$ being such as to cause the disappearance of the periodical term

$$\psi'_o(u'U' \dots)_1 \tag{77}$$

by reducing the expression

$$\psi_o(u' \dots)_o + \psi_o(u'U' \dots)_1 \tag{78}$$

to the constant term

$$\psi'_o(u' \dots)_o \tag{79}$$

Hence

$$\begin{aligned} u &= \delta_o u_1 + u' \\ U &= \delta_o U_1 + U' \end{aligned} \tag{80}$$

and

$$\begin{aligned} u' &= u + \Sigma_2^\infty \delta_o u_n \\ U' &= U + \Sigma_2^\infty \delta_o U_n \end{aligned} \tag{81}$$

By repeated operations of this kind, Delaunay finally reduces his expression to a stage where the variables are

$$\begin{aligned} u^i &= u + \Sigma_{i+1}^\infty \delta_o u_n \\ U^i &= U + \Sigma_{i+1}^\infty \delta_o U_n \end{aligned} \tag{82}$$

and the disturbing function has the form

$$R_o^i = M_o \psi_o^i(u^i \dots)_o + M_o \Sigma_{i+1}^\infty \psi_o^i(u^i U^i \dots)_n \tag{83}$$

Delaunay then assumes that all the terms in the perturbations beyond $u_i\ U_i$ are so small that they may be regarded as insensible, so that the terms denoted by

$$\Sigma_{i+1}^\infty \psi_o^i(u^i U^i \dots)_n \tag{84}$$

may be neglected. Hence his expressions reduce to the form

$$\begin{aligned} u^i &= u \\ U^i &= U \end{aligned} \tag{85}$$

which are constant quantities, and can be determined from the expression for the disturbing function now reduced to the non-periodical term

$$R_o^i = M_o \psi_o^i(u \dots)_o \tag{86}$$

By these successive operations Delaunay has thus successively determined every term in the complete value of the perturbations

of the elements, whose complete values are now expressed by the known series

$$\begin{aligned} \mathrm{u} &= u + \Sigma_1^{\infty} \delta_0 u_n \\ \mathrm{U} &= U + \Sigma_1^{\infty} \delta_0 U_n \end{aligned} \tag{87}$$

There can be no objection to the principles of this method of determining the perturbations of purely solar origin, or to the admirable manner in which it has been carried out by Delaunay in his "Théorie de la Lune."

Its extension, when it becomes necessary to take into account the perturbation arising from the disturbing action of a planet, is effected by Delaunay in the following manner.

The complete values of the elements of the instantaneous orbit of the Moon become now, for convenience changing the letter used,

$$\begin{aligned} \mathrm{v} &= v + \Sigma_1^{\infty} \delta_0 v_n + \Sigma_0^{\infty} \delta_1 v_n \\ \mathrm{V} &= V + \Sigma_1^{\infty} \delta_0 V_n + \Sigma_0^{\infty} \delta_1 V_n \end{aligned} \tag{88}$$

where the additional series of terms are

$\Sigma_0^{\infty} \delta_1 v_n$ $\Sigma_0^{\infty} \delta_1 V_n$ denoting the series of terms depending on the disturbing action of the planet.

The complete value of the disturbing function is now

$$\mathrm{R} = \mathrm{M}_0 \Sigma_0^{\infty} \psi_0 (\mathrm{v}\ \mathrm{V} \ldots)_n + \mathrm{M}_1 \Sigma_0^{\infty} \psi_1 (\mathrm{v}\ \mathrm{V} \ldots)_n \tag{89}$$

where M_1 denotes the mass of the disturbing planet.

As before, Delaunay proceeds to transform the variables v V to the new variables v′ V′, such that

$$\begin{aligned} \mathrm{v}' &= \mathrm{v} - \delta_0 v_1 \\ \mathrm{V}' &= \mathrm{V} - \delta_0 V_1 \end{aligned} \tag{90}$$

where, as before,

$$\delta v_1 \ \delta V_1$$

are so determined that when the new variables are substituted in R they cause the disappearance of the periodical term with the argument 1, so that the new value of the disturbing function has the form

$$\mathrm{R}' = \mathrm{M}_0 \psi'_0 (\mathrm{v}' \ldots)_0 + \mathrm{M}_0 \Sigma_2^{\infty} \psi'_0 (\mathrm{v}'\ \mathrm{V} \ldots .)_n + \mathrm{M}_1 \Sigma_0^{\infty} \psi'_1 (\mathrm{v}'\ \mathrm{V}' \ldots)_n \tag{91}$$

Hence, as before,

$$\begin{aligned} \mathrm{v} &= \delta_0 v_1 + \mathrm{v}' \\ \mathrm{V} &= \delta_0 V_1 + \mathrm{V}' \end{aligned} \tag{92}$$

and

$$\begin{aligned} \mathrm{v}' &= v + \Sigma_2 \ \delta_0 v_n + \Sigma_0^{\infty} \delta_1 v_n \\ \mathrm{V}' &= V + \Sigma_2^{\infty} \delta_0 V_n + \Sigma_0^{\infty} \delta_1 V_n \end{aligned} \tag{93}$$

By repeating the same series of operations as before, Delaunay reaches, at last, the same stage where he ended the transformations in the case of the perturbations due solely to the action of the Sun, the stage where the variables have been reduced to

$$\begin{aligned} v^i &= v + \Sigma_{i+1}^{\infty} \delta_0 v_n + \Sigma_0 \delta_1 v_n \\ V^i &= V + \Sigma_{i+1}^{\infty} \delta_0 V_n + \Sigma_0^{\infty} \delta_1 V_n \end{aligned} \tag{94}$$

and the disturbing function has the form

$$R^i = M_0 \psi_0^i (v^i \ldots)_0 + M_0 \Sigma_{i+1}^{\infty} \psi_0^i (v\ V\ \ldots)_n + M_1 \Sigma_0^{\infty} \psi_1^i (v^i V^i \ldots) \tag{95}$$

Then, by neglecting as insensible the same quantities as before, Delaunay reduces these expressions (94) and (95) to

$$\begin{aligned} v^i &= v + \Sigma_0 \delta_1 v^n \\ V^i &= V + \Sigma_0^{\infty} \delta_1 V_n \end{aligned} \tag{96}$$

and

$$R_i = M_0 \psi_0^i (v \ldots)_0 + M_1 \Sigma_0 \psi_1^i (v\ V \ldots)_n \tag{97}$$

whilst the original variables take the form

$$\begin{aligned} v &= v + \Sigma_1^i \delta_0 v_n + \Sigma_0^{\infty} \delta_1 v_n \\ V &= V + \Sigma_1^i \delta_0 V_n + \Sigma_0^{\infty} \delta_1 V_n \end{aligned} \tag{98}$$

the last terms of these expressions (98) being still so far arbitrary, inasmuch as they have not yet received determinative values.

The preceding shows clearly the manner in which Delaunay reduces the value of the first term of the complete disturbing function (89) from

$$R_0 = M_0 \Sigma_0^{\infty} \psi_0 (v\ V \ldots)_n \tag{99}$$

to the value

$$R_0^i = M_0 \psi_0^i (v \ldots)_0 \tag{100}$$

which he employs as the value of this portion of R to be used in the differential equations (59) and (61), so as to determine the complete value of the perturbation arising from the disturbing action of the planet, denoted here by

$$\Sigma^{\infty} \delta_1 v_n \qquad \Sigma_0^{\infty} \delta_1 V_n$$

The question is now, has this reduction from (99) to (100) been legitimately effected, without omitting terms which ought to have been retained?

To this I reply, 'No!'

Compare the two series of transformations (75) and (90). In the first we have the value

$$u' = u - \delta_0 u_1$$

in the second the value

$$v' = v - \delta_0 v_1$$

Now, throughout his investigation Delaunay assumes the complete identity of the two series

$$\Sigma_1^{\infty}\ \delta_0\ u_n = \Sigma_1^{\infty}\ \delta_0\ v_n$$

so that we have

$$v' = v - \delta_0\ u_1 \tag{101}$$

and so by (73) and (88)

$$v = u + \Sigma_0^{\infty}\ \delta_1\ u \tag{102}$$

the preceding expression (101) becomes

$$v' = u - \delta_0 u_1 + \Sigma_0^{\infty} \delta_1 v_n \tag{103}$$

Hence

$$v' = u' \left\{ 1 + \frac{\Sigma_0^{\infty} \delta_1 v_n}{u'} \right\} \tag{104}$$

Now the entire validity of Delaunay's method of calculating the perturbations of this planetary origin rests on the assumption made in passing from (89) to (91), that as the operation of transforming the variables from u U to u′ U′ reduces the periodical expression

$$\psi_0(u' \ldots\ldots)_0 + \psi_0(u'U' \ldots\ldots)_1$$

to the non-periodical expression

$$\psi'_0(u' \ldots\ldots)_0$$

(77) to (79), so will *identically* the same operation with the variables v V and v′ V′ reduce the periodical expression

$$\psi_0(v' \ldots\ldots)_0 + \psi_0(v'V' \ldots\ldots)_1$$

to the non-periodical term

$$\psi'_0(v' \ldots\ldots)_0$$

Hence, Delaunay's entire method rests on the assumption that because

$$\psi_0(u' \ldots\ldots)_0 + \psi_0(u' . U' \ldots\ldots)_1 - \psi'_0(u' \ldots\ldots)_0 = 0 \tag{105}$$

so must

$$\psi_0(v' \ldots\ldots)_0 + \psi_0(v'V' \ldots\ldots)_1 - \psi'_0(v' \ldots\ldots)_0 = 0 \tag{106}$$

For simplicity, put

$$\begin{aligned} j &= \Sigma_0^{\infty} \delta_1 v_n & J &= \Sigma_0^{\infty} \delta_1 V_n \\ D_j &= \frac{d}{u' du'} & D_J &= \frac{d}{U' dU'} \end{aligned} \tag{107}$$

then by (104), expanding by a symbolical form of Taylor's Theorem

$$\psi_0'(v' \ldots\ldots)_0 = \left\{ \epsilon^{jD_j} \right\} \psi_0'\{u' \ldots\ldots)_0$$

$$\psi_0(V' \ldots\ldots)_0 = \left\{ \epsilon^{jD_j} \right\} \psi_0(u \quad \ldots\ldots)_0 \qquad (108)$$

$$\psi_0(v'\, V' \ldots\ldots)_1 = \left\{ \epsilon^{jD_j + JD_J} \right\} \psi_0(u'U' \ldots\ldots)_1$$

Hence (106) may be written in the form

$$\left[\epsilon^{jD_j}\right] \left\{ \psi_0(u' \ldots\ldots)_0 + \psi_0(u'U' \ldots\ldots)_1 - \psi_0'(u' \ldots\ldots)_0 \right\}$$
$$+ \left[\epsilon^{jD_j}\right] \left[\epsilon^{JD_J} - 1\right] \left\{ \psi_0(u'U' \ldots\ldots)_1 \right\} = 0 \qquad (109)$$

By (105) the first term of this is equal to zero, hence Delaunay's assumption reduces to the condition that

$$\left[\epsilon^{jD_j}\right] \left[\epsilon^{JD_J} - 1\right] \left\{ \psi_0(u'U' \ldots\ldots)_1 \right\} = 0 \qquad (110)$$

and unless this condition hold generally true, Delaunay's method fails, for he has omitted the entire series of terms represented by the function on the left-hand side of (110).

By replacing by their known equivalents the function

$$\epsilon^{jD} \qquad \epsilon^{JD_J}$$

the value of the left-hand side of (110) may be written in the form

$$\left\{ \left[1 + \Sigma_0^\infty \delta_1 v_n \cdot \frac{d}{u'du'} \cdot + \ldots\ldots \right] \left[\Sigma^\infty \delta_1 V \cdot \frac{d}{U'dU'} + \ldots\ldots \right] \right\}$$
$$\psi_0 (u'\, U' \ldots\ldots)_1 \qquad (111)$$

It is evident that this expression will represent a series of periodical terms which cannot vanish, either invariably or even generally; and they will thus form an important portion of the transformed value of R, which has been uniformly neglected by Delaunay. Thus the reduction from (99) to (100) has only been effected by neglecting an important portion of the complete value of the disturbing function.

It would not be difficult to show, were it necessary, that the terms represented by (111), which have been omitted by Delaunay, correspond exactly with those represented by the third portion of the disturbing function denoted by R''' in my developments (19) and (69), which yield the second portion of the equation (72). And their omission by Delaunay explains the discordance between the results obtained by the two methods.

It follows, therefore, that the method employed by Delaunay does not yield the complete value of the coefficients of equation of long period due to the disturbing action of the planet.

The importance of this result is evident.

First, there is thus explained the discordance between the

results obtained by myself by the methods which I have explained above, and the results obtained by other astronomers for the same terms of long period by means of Delaunay's method.

Second, what is far more important is its results with regard to the comparison between theory and observation. My own researches have shown clearly that if Delaunay's values for these terms be accepted as accurate, it is impossible to reconcile the theory of the Moon with the results of actual observation without assuming the existence of a large inequality of long period, which can only be due to the disturbing action of a planet. Now, unless Delaunay's method of determining the value of these terms be incorrect, there cannot exist such an inequality. Every one which could possibly yield a sensible coefficient by Delaunay's method has been carefully examined by me. They number over one hundred, but not a single one can have a maximum value of nearly one second, whereas to reconcile theory and observation there is required a term with a coefficient of twelve seconds (*On the Corrections to Hansen's Tables de la Lune. Memoirs R. A. S.*, vol. xlviii. p. 372). Hence an apparently unsurmountable discordance between theeory and observation.

But Delaunay's method now having been shown to be imperfect, there are two methods at once available for meeting this difficulty. First, as I have shown already (*ibid.*, page 371), by assuming smaller and more probable values for the coefficients of Hansen's two Venus terms to those employed by him in his tables. This suffices to remove by far the greater part of the existing discordance, and the coefficients of these terms thus indicated by observation agree fairly well with the results of my approximate theoretical computations. Second, by rendering it possible that the whole remaining discordance between the results of theory and observation may be removed ultimately by the calculation of the values of the terms of long period which are known to exist; for though it is evident that the coefficients of these terms will be too small to be of any importance when they are calculated by Delaunay's method, it may be anticipated that many of these coefficients will be found to be much larger when calculated by one of the complete methods that have been described in the first portion of this Paper, so that some may be found large enough to remove the outstanding discordances.

In conclusion, perhaps it will be as well to add a few words on the exact condition of my own investigation of these terms. They have been carried out quite independently by both of the two methods already described, namely, the method of disturbed co-ordinates, and the method of perturbed elements. They are already so far advanced as to only require the calculation of some supplementary terms to be prepared by final revision for reduction to numbers, so as to enable me to use them to obtain complete results. A considerable amount of additional work, however, must necessarily be involved in preparing such a memoir for

publication. Ill-health, unfortunately, has completely prevented my touching them for nearly a year, but I hope that before the end of this year I shall be able again to resume my labours, and bring them to an early close. The work is one of very great length, as many of the coefficients have had to be carried to the tenth order of small quantities. I do not hesitate to say that it involves as much work as is necessary to carry to the seventh order the calculation of the ordinary lunar perturbation of purely solar origin, such as form the contents of the two great volumes constituting Delaunay's "Théorie de la Lune." For it is to be remembered that by far the greatest portion of the labour has been neglected by those astronomers who have employed Delaunay's method, and who have in consequence had to face scarcely one-tenth of the work necessary to calculate the complete value of the coefficients of these terms of long period due to the disturbing action of the planets.

Moreover, the work which is being carried out by myself embraces in one plan the computation of the entire series of nearly fifty terms of long period due to the disturbing action of the different planets, which I have reason to believe have coefficients of sufficient magnitude to render it advisable that they should be retained in any future lunar tables. It would be most inconvenient, therefore, and would involve much additional labour, for me to have to select from the whole mass of computations, and to piece together in a fit state for publication, the numerous detached portions of the whole which relate to those particular terms of long period. And if the accuracy of my values for their coefficients has been questioned by other astronomers, it has been on the basis of investigations which I have shown, in the present Paper, to rest on an imperfect method.

I think, therefore, it is the more preferable course to wait until I have completed the whole investigation, and am in a position to publish entire, my memoir on these important perturbations in the motions of the Moon.

The Natal Observatory,
May 1, 1886.

Supplementary Measures of the Magnitudes of a Zone of Stars near the Equator for reference as Standards of Magnitude in lieu of Polaris. By Prof. C. Pritchard, D.D., F.R.S.

In the course of the last twelve months I have had the photometric remeasurement of about fifty stars near the equator effected. These stars are of approximately the magnitudes four and six, and may well serve as stars of comparison for other stars, whether north or south of the equator, and may be used for such comparison within any magnitudes from the first to the ninth without reference to *Polaris* when such reference is incon-

venient. Moreover, in comparing the magnitude of southern stars, the uncertain correction for atmospheric absorption may be avoided, and the relative brightness of stars in the southern hemisphere can be connected with those already determined in the north, systematically, and on the same scale.

In the text of the "Uranometria Nova Oxoniensis" the magnitudes of the stars in this zone have been determined for the most part from twenty extinctions on one night. In the present supplement, the results have been obtained from measures made on at least five nights, so that the concluded magnitudes now depend on, in general, seventy extinctions. I think, therefore, that these magnitudes are worthy of very considerable confidence. In the supplementary table here presented, the fourth column contains the magnitude already given in the "Uranometria Nova Oxoniensis," the second is the concluded magnitude derived from all the extinctions of the several nights of observation. It is, I think, highly satisfactory to find that no important corrections are to be made to the original determinations.

Star's Name.	R.A. 1890.	N.P.D. 1890.	Old determination of Magnitude in "Uran. Nov. Ox."	New determination of Magnitude 1885–86.
	h m	° ′		
P. o.146	o 35·1	94 57	6·25	6·19
δ Piscium	o 43·0	83 1	4·32	4·43
88 Piscium	1 9·0	83 35	6·01	6·03
ν Piscium	1 35·7	85 4	4·68	4·65
P. ii. 52	2 12·3	88 46	6·02	5·97
ξ^2 Ceti	2 22·3	82 2	4·66	4·56
95 Ceti	3 12·7	91 20	5·77	5·72
κ Ceti	3 13·6	87 2	5·15	5·04
40 Tauri	3 58·0	84 52	5·56	5·51
P. iv. 49	4 14·9	84 8	6·02	6·02
Lal. 9579	4 59·4	93 12	6·00	6·05
π^6 Orionis	4 52·9	88 27	4·74	4·70
ψ^1 Orionis	5 19·0	88 15	4·61	4·57
P. v. 206	5 39·2	86 2	6·09	6·10
D.M. +2° No. 1237	6 21·6	87 1	6·00	6·00
10 Monocerotis	6 22·5	94 42	4·82	4·82
13 Monocerotis	6 27·0	82 35	4·78	4·59
Lal. 13799	7 1·9	82 21	6·06	6·05
β Can. Min.	7 21·2	81 29	3·11	3·10
Lal. 15136	7 40·7	96 30	5·98	5·96

Star's Name.	R.A. 1890.	N.P.D. 1890.	Old determination of Magnitude in "Uran. Nov. Ox."	New determination of Magnitude 1885–86.
	h m	° ′		
30 Monocerotis	8 20·2	93·33	3·63	3·64
D.M.—1° No. 2125	8 40·4	91 39	...	6·06
21 Hydræ	9 7·0	96 40	6·20	5·99
ι Hydræ	9 34·2	90 39	4·36	4·37
12 Sextantis	9 54·0	86 5	6·73	6·67
23 Sextantis	10 15·4	87 9	6·79	6·74
30 Sextantis	10 24·7	90 4	5·02	5·07
55 Leonis	10 50·1	88 41	6·20	6·10
σ Leonis	11 15·5	83 22	4·15	4·13
b Virginis	11 54·3	85 44	5·36	5·32
χ Virginis	12 33·6	97 23	4·67	4·67
δ Virginis	12 50·1	86 0	3·47	3·49
46 Virginis	12 54·9	92 47	5·97	5·94
ζ Virginis	13 29·2	90 2	3·36	3·33
80 Virginis	13 29·8	94 50	5·91	5·89
W.B. xiv. 143	14 10·8	92 41	6·02	6·03
μ Virginis	14 37·3	95 11	3·86	3·85
Lal. 27297	14 53·9	84 59	6·31	6·30
ψ Serpentis	15 38·5	87 8	5·83	5·84
μ Serpentis	15 43·9	93 6	3·30	3·33
21 Herculis	16 18·7	82 48	5·94	5·95
κ Ophiuchi	16 52·3	80 28	3·17	3·22
P. xvi. 289	16 59·9	90 45	5·68	5·62
γ Ophiuchi	17 42·4	87 15	3·83	3·78
W.B. xviii. 28	18 4 4	86 53	5·86	5·78
Lal. 35281	18 50·7	91 56	6·05	6·08
δ Aquilæ	19 19·9	87 6	3·36	3·33
σ Aquilæ	19 33·8	84 51	4·88	4·77
71 Aquilæ	20 32·7	91 29	4·63	4·57
7 Pegasi	21 36·8	84 49	5·41	5·41
28 Aquarii	21 55·4	89 56	5·85	5·75
η Aquarii	22 29·7	90 41	4·12	4·14
3 Piscium	22 55·0	90 24	6·48	6·38
ι Piscium	23 34·3	84 58	4·25	4·27
21 Piscium	23 43·8	89 32	6·11	6·09

Since the completion of the "Uranometria Nova Oxoniensis" the method of stellar photography has been notably improved, mainly by the methods adopted and set forth by Mr. Common in England and by the Messrs. Henry at Paris.

For my own part, I have, at the Oxford University Observatory, studied this method with, I trust, considerable success, and the several results of this research have been communicated to the Royal Society. I mention this fact here, because I believe it will be found that the photographic method, properly applied, is pre-eminently adapted to the photometry of the fainter stars, including all fainter than the seventh magnitude.

The relation which in these researches is established between the photographic diameters of the discs of stars impressed on the *same plate* during one exposure of the film is

$$D - D' = \delta\{\log M' - \log M\}$$

where D and D′ are the measured diameters of the discs of any two stars and M, M′ their corresponding magnitude, δ being a constant determined from that plate. The preceding formula when translated into words indicates that, while the numerical magnitude increases in geometrical progression, the diameter diminishes in arithmetic. The magnitudes thus obtained are, strictly speaking, photographic; but for the great majority of stars possessing no peculiarity in their actinism or spectra, these photographic magnitudes correspond with the photometric.

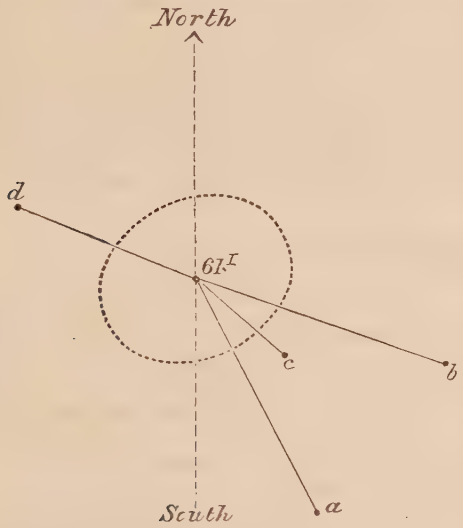
In practice it is necessary to measure photometrically the magnitudes of three or four out of the stars impressed on the plate, when the formula above will give the magnitudes of the remaining stars, after measuring the diameters of their discs, either with a double-image micrometer or by any other suitable means. (See "Proceedings of the Royal Society" for May 1886.)

On a Remarkable Instance of the Detection of Distortion in a Photographic Film, measured for the purpose of Stellar Parallax. By Prof. C. Pritchard, D.D., F.R.S.

In the new and important relations recently established between photography and observational astronomy, any communication, however trifling, relating thereto, so long as it is accurate and sincere, may be acceptable to the Society. In this view I enclose an account of a somewhat remarkable condition, in which I have twice found a photographic film, which at once conveys a warning and encouragement in the use of stellar photography when applied to precise instrumental measurement.

The case here in question refers to photographs of the two stars, 61^1 and 62^2 *Cygni*, and neighbouring stars. The object is

the somewhat ambitious, and maybe hazardous, attempt to extract (as a first trial of a new method) the parallax of this well-known star from the careful measures of the distances of four stars of comparison from the two components of 61 *Cygni*. The following is a diagram, drawn to scale, of the stars in question. The dotted ellipse indicates the approximate form and position of the parallactic ellipse, of course enormously enlarged. The plates are usually exposed for not more than five



minutes, the object being to limit the diameter of the stellar discs to the smallest dimensions compatible with distinct impressions of the fainter comparison stars on the film. With much longer exposures the stellar discs would be more difficult in their instrumental bisection. Moreover, the two discs of 61 *Cygni* might probably coalesce. The circumstances to which I desire to call attention are these:

On each of eight plates, taken on four contiguous nights, the distances of the stars *a* and *b*, from 61^1 and 61^2 *Cygni* have been measured, as also the distances of the stars *c* and *d* from the two components. These measures, as regards seven of the plates, exhibit very satisfactory agreement *inter se* when fully reduced for refraction and aberration, presenting a probable error not exceeding the tenth of a second of arc, promising so far, but so far only, the ultimate success of the research. But on one of these plates there is this remarkable fact, viz. the distances of the stars *a*, *b* and *c* from 61^1 and 61^2 are each definitely less (by about $1''\cdot2$) than the corresponding measures on all the other seven plates. This would of itself indicate a disturbance of the film in the neighbourhood of either 61 or of

a, *b*, *c*. The fact of this disturbance on the eighth plate is not only confirmed, but is actually localised by the measures of the star *d*, from 61^1 and 61^2, for this measure is in good accordance with the similar measures on all the other seven plates.

The film has therefore been disturbed in the neighbourhood of *a*, *b*, *c*, and remained unimpaired in other parts. From these results I think we may fairly deduce the reliability of the whole process, both in regard to the ultimate precision, and the power of detecting flaws in the photographic film. It may be well to state that the delicacy of the measures does not extend beyond the hundred-thousandth of an inch on the film, amounting to about 0″·07. The plates are purposely confined to two inches square, embracing the limit of accuracy of the telescopic field, whilst beyond these dimensions the photographs might resemble pictures rather than accurate charts, capable of exact measurement.

Oxford University Observatory,
1886, *June* 10.

On the Orbit of ζ Sagittarii. By J. E. Gore.

This bright southern star was discovered to be double many years ago by Winlock with the 15-inch Refractor of the Harvard Observatory. Very few measures of it have been made, but those by Burnham in 1878–1881 show that it is undoubtedly a binary in very rapid motion.

I have computed the orbit by the graphical method described in the "Handbook of Double Stars" by Messrs. Crossley, Gledhill, and Wilson.

The observations used are as follows:—

Epoch.	Position Angle.	Distance.	Observer.
1867·8	260°·8	0″·48	Newcomb
1878·70	84·2	0·42	Burnham
1880·62	62·1	0·55	,,
1881·61	36·1	0·31	,,

From these I have computed the following provisional elements:

Elements of ζ Sagittarii.

P = 18·69 years	Ω = 83° 22′
T = 1882·86	λ = 263° 21′
e = 0·1698	a = 0″·53
γ = 58° 48′	μ = −19°·26

The observations are, of course, much too few to enable a satisfactory orbit being computed, but I think the observed measures

show that the period found above cannot be very far from the truth. With the exception of *ε Equulei*, it seems to be the shortest period yet computed for a binary star.

The following is a comparison between the recorded observations and the positions computed from the above elements :—

Epoch.	θ_o	θ_c	$\theta_o-\theta_c$	ρ_o	ρ_c	$\rho_o-\rho_c$
1867·8	260°·8	260°·2	+0°·6	0″·48	0″·51	−0″·03
1878·70	84·2	85·0	−0·8	0·42	0·53	−0·11
1880·62	62·1	61·4	+0·7	0·55	0·40	+0·15
1881·61	36·1	38·2	−2·1	0·31	0·29	+0·02

Owing to the closeness of the components, and the great southern declination of the star, it cannot be well measured in this country. It is therefore to be hoped that observers in the Southern Hemisphere will pay some attention to this interesting binary during the autumn of 1886 and 1887. If the above elements are correct, the position-angle and distance must at present be nearly the same as when the star was measured in 1867, a complete revolution having been performed in the interval, and no more favourable opportunity of measuring it will occur till the year 1896.

The position of the star is for 1886·0

R.A. $18^h\ 55^m\ 21^s$

S. 30° 3′ 20″

The formulæ for calculating an ephemeris are

$$u-9\cdot728\sin u=-19°\cdot26\,(t-1882\cdot86)$$

$$\tan\tfrac{1}{2}V=1\cdot19\tan\tfrac{1}{2}u$$

$$\tan(\theta_c-83°\ 22')=0\cdot5180\tan(V+263°\ 21')$$

$$\rho=0''\cdot53\,(1-0\cdot1698\cos u).\frac{\cos(V+263°\ 21')}{\cos(\theta_c-83°\ 22')}$$

where u is the eccentric anomaly, and V the true anomaly for the time t, θ_c the required position-angle, and ρ the distance.

The magnitudes of the components are about 3·5 and 4·0.

1886, *May* 25.

The Northern Hemisphere of Mars. By N. E. Green.

It has been considered advisable to re-examine the details of the northern hemisphere, in order to correct or improve the map of *Mars* published in connection with the Madeira Series in 1877, that portion of the map being formed from drawings made in 1873, and considered provisional.

A careful study of the northern markings was made by Mr. Knobel in 1884, and during the recent opposition the writer has

followed the planet with renewed attention. The result of these observations has been the confirmation of many of the old forms and the filling in of details over an extensive portion of the surface lying between longitude 90° × 180°, which part was nearly a blank in the map of 1877.

Some differences exist between that map and the one produced by Mr. Knobel from his observations in 1873 and 1884, to which special reference is made in this paper.

The instrument employed in the recent observations is an 18-inch Reflector; the speculum was figured by Mr. G. With of Hereford, and is of the highest quality. The powers employed varied between 280 and 560, the latter being used most frequently. The details of the northern hemisphere of *Mars* require steady air and great amplification in order to form a true estimate of their shapes, and it was found that the more delicate markings could not be drawn without the assistance of a power of 400. Many forms of eyepiece were tried, but that which gave the best results was a binocular of the usual microscopic shape, screwed to a tube connected with the adjusting rack. As this binocular eyepiece gives an erect view, it was considered better to complete a series of drawings in this new position, without comparing them with previous views till they were finished; and it was very interesting to observe afterwards how the forms which at the telescope appeared new and strange fell into their places, and in many instances agreed closely with the map of 1877. A copy of this map, together with a new one of the northern portion, produced from the drawings made this year, accompanies the paper.

On comparing these maps, it will be seen that from longitude 0 to 60° there is but little change, though Airy Sea is now a less definite form. From 90° to 180° an entirely new set of markings appear, some of them strongly resembling the light mark called Fontana Land at longitude 210°. These shapes were singularly definite, and of the truth of their forms and colour the observer has no doubt.

Huggins Inlet, passing from Maraldi Sea across the equator towards Fontana Land, has been observed on several occasions, but appeared as a broader and fuller mark than in the views by Mr. Knobel and others.

From longitude 210° to 300° there is no definite change. The termination of Nasmyth Inlet, called Lassell Sea, has been a most prominent mark during the opposition just past, but less extended towards the equator than as drawn in the older map. The light space called Leverrier Land has been seen as a clear separation between Lassell and Knobel Seas. It is at this portion of the map Mr. Knobel thinks that corrections should be made, but the most close and continued observation has failed to discover any decisive difference of shape between the forms observed in 1873 and during the past opposition.

It is but just to remark that in 1884 a drawing was made of

the termination of Nasmyth Inlet which agrees exactly with the description of its shape at page 280 of Mr. Knobel's Memoir; and certainly during that opposition there was no approach to the definite rounded form seen this year.

I am indebted to Mr. Denning of Bristol and Mr. Smart of London for several views of *Mars*; and the drawings of both these observers confirm the existence of Lassell Sea as a roundish spot.

One of the most marked features of the opposition now past has been the frequent appearance of masses of light on the limb which never arrive at the meridian, and that the orange portions of the surface so frequently seen on the meridian do not retain this colour as they pass towards the limb, but become whitish. May not this suggest that cloud condensation prevails on the right side of the planet, and that these masses of cloud are dispersed as they pass the meridian towards the Sun?

The character of the markings on *Mars*, the difficulty of observing, and still more the difficulty of drawing them, will continue to make this planet, if not a satisfactory, at least a most interesting, object of study.

On the asserted Foreshortening of the Inner Side of the Penumbra of Spots when near the Sun's Limb. By Rev. F. Howlett.

I am desirous of placing on record some definite statements and measurements in connection with the appearance of certain symmetrically shaped solar spots, both when near the centre and also when close to the limb of the Sun.

It will be found that these measurements entirely militate against the commonly received opinion that the spots are to any such extent sunk in the solar surface as to produce always those effects of perspective foreshortening of the inner side of the penumbra (when near the limb) which have been described in various works on Astronomy, and founded chiefly, I believe, on statements made as long ago as the year 1769 by Dr. Wilson of the University of Glasgow.

These statements and observations on the part of Dr. Wilson are so extremely circumstantial that I am far from desiring to call them in question, though I beg to be allowed to affirm that they at any rate do not always hold good.

Dr. Wilson's statements are thus alluded to in Mr. Proctor's very full and interesting popular work *The Sun*, ed. 1871, p. 169. "In November 1769 Dr. Wilson began the careful study of a large spot (visible to the naked eye). . . . When he first examined the spot (Nov. 22) it was situated not very far from the western or 'preceding' edge of the Sun's disk.

"On the next day he found that the spot had changed its appearance. The penumbra, which on the previous day was

equally broad on all sides of the nucleus [or as we should now say, more precisely, the umbra], was now very much contracted on the side which lay towards the centre of the disk, while the other parts retained nearly their former dimensions. On the 24th he again observed the spot. The distance from the limb was now only twenty-four seconds, and the contracted side of the penumbra had entirely vanished. . . . The breadth of the nucleus [umbra] on the same side also appeared to be more suddenly impaired than it ought to have been by the motion of the Sun (*sic*) across the disk.

"Dr. Wilson showed that these results correspond with those which would follow if the spot was a vast cavernous opening, having the nucleus at the bottom, and the penumbra forming its sloping sides.

"It only remained to be seen whether a corresponding succession of changes occurred when the spot reappeared on the eastern edge, and thence passed across the solar disk. This actually happened. On December 11 the spot appeared on the opposite side of the disk. The side of the penumbra next the edge, which formerly vanished, was now wholly visible, while that turned towards the centre of the disk appeared to be wanting. On December 12 it [that is, this part of the penumbra] came into view, and he saw it distinctly, though narrower than the other side. He did not see the spot again till December 17, when it had passed the centre of the disk, and the penumbra appeared to surround the nucleus [umbra] equally on all sides."

Not, however, that such statements, however well attested on the above occasion, have always remained unchallenged. For instance, on the occasion of the eleventh General Assembly of the Astronomical Society, convened at Geneva in the month of August 1885, Professor Spörer of Munich, in a copious *résumé* of his labours in connection with the solar spots, denies that the spots possess the character of funnels (*tonnoirs*), which is attributed to them in the greater number of works on elementary astronomy. The appearances which attend the disappearance of spots on the Sun's western limb cannot, he affirms, be explained by such a theory. "When a spot," he says, "is very shortly about to disappear, by virtue of the Sun's rotation on his axis, the two *lateral* borders of the penumbra vanish from sight, whilst the nucleus [umbra] remains visible, together with the fragments of the surroundings (*pourtour*) to the north and south; the whole phenomena," he says, "presenting the appearance of a veiled brightness (*éclat voilé*) to such a degree that one might easily confound the nucleus [umbra] itself with its surroundings," and of which typical and striking instances may be seen in vol. ii. sheet 99, and still more sheet 108, of my drawings.

Spörer then seeks to explain this appearance by a heated facular border surrounding the spot, and causing an ascending current of extremely heated gas, through which it is with the

greater difficulty that the spot is discerned the nearer it is to the limb.

He then remarks, very justly, that it is "by no means the case that all spots are suitable for the estimation of the pretended parallax of the depth of a spot; and in making use of documents on this subject one is far from arriving at results similar to some of those which have been published; and that in all attempts to make drawings of the spots there is a danger of falling into grave illusions."

In my contention, again, that by no means the majority of spots present that funnel-like appearance attributed to them by the Wilson hypothesis, I am sufficiently borne out both by Father Perry and Mr. Ranyard. The former, after paying especial attention, at my request, to this point in connection with a large and unusually symmetrical spot which was close to the western limb on the afternoon of April 12 of this present year, wrote me thus:—"The question of the foreshortening was specially attended to, and the observation showed a very slight excess of foreshortening on the side of the penumbra nearest the Sun's centre, but certainly not enough to give any support to Wilson's theory." Mr. Ranyard, again, in a letter of the 7th inst., reports:—"I had a good look at your spot on the morning of Wednesday, June 2, and I again had a search for it on the morning of June 4" (when, I may observe, it had quite passed off the limb). "I quite agree with you," he says, "that there was no appreciable difference in the breadth of the penumbra on the preceding and following sides of the nucleus" (umbra, again, is here meant) "when I observed it on the morning of Wednesday, June 2. I was observing it between 9^h 20^m and 10^h A.M., and I thought that the penumbra was a little *darker* on the preceding than on the following side, but not very noticeably so. There was no band of increased darkness such as would have been produced by a column of absorbing matter rising vertically above the nucleus. I quite agree with you that the phenomena observed did not tend to support Wilson's hypothesis."

With respect to this last spot, Father Perry's observations do not confirm my own to the same extent, apparently, as was the case in connection with the spot γ of April last, though it is on that of June 2 that I would rely, if possible, more firmly than on the rest. Together with some small but beautifully executed copies of the sketches of spots made recently at Stonyhurst, Father Perry wrote as follows:—"You will notice that near the edge there appears to be a slight confirmation of Wilson's theory at both limbs of the Sun." And he adds, "In some of our drawings of solar spots this is much more marked; but these probably are different classes of spots, and all certainly do not behave alike."

But now as to actual *measurements* of what I would term the crucial spots γ of April 12, and λ of June 2.

I first noticed γ on April 2, at 7^h 30^m A.M., when it was about

2′ 48″ from the eastern limb, and perceiving that the umbra was nicely central, and the whole spot neatly oval, I resolved to keep the spot under observation, with the view of testing the hypothesis in question. It was then about 50″ in length from north to south, and, as then, foreshortened on the sphere, about 40″ in breadth from east to west. On April 6 the size of the spot was somewhat diminished, but its neatness and suitability for testing the hypothesis was still more apparent, for a more perfectly central umbra or symmetrical penumbra could not be wished for. The spot at $10^h\ 20^m$ A.M. that day was not far from the centre of the disk, and subtended about 45″ in length and the same in breadth. When at length, on April 12, the spot had arrived very near to the western limb, I made the following notes: At $4^h\ 20^m$ γ was only about 10″ from the western edge of the Sun, and subtended not more than 10″ in breadth. The umbra still appeared quite central, with a width of not more than 2″. Each side of the penumbra measured also about 2″ in breadth, or possibly the "preceding," or outer, side, 2″, and the following 1″·8, but of this I could not be quite certain.

I may observe also, in passing, that on May 9 a spot lettered ι, when about 18″ only from the western limb, and measuring 40″ in length and 8″ in breadth, presented a perfectly central umbra 3″ in width; here again militating against Dr. Wilson's hypothesis.

But now for the spot λ of June 2, 1886. This was also a remarkably symmetrical one, of a large size, and specially adapted to the investigation in hand. On June 1 the spot was, at $5^h\ 30^m$ P.M., just one minute from the western limb, yet there was not the slightest appearance of any foreshortening of the *inner* side of the penumbra. In fact, if there was a difference, the advantage was by about 1″ or so on the *inner* side. The spot now subtended about 45″ in length, by about 17″ in breadth.

On June 2 (the day on which I had invited Mr. Ranyard, and also again Mr. Perry, to specially watch for Wilson's phenomenon) I made these notes: At $7^h\ 50^m$ A.M. the outer border of the penumbra of the spot λ was just 20″ from the western limb. The whole spot subtended 45″ in length of heliographical latitude, but so foreshortened in directions of longitude as to subtend not more than 10″ in width.

The *umbra* was as nearly as possible central, and having an apparent width of 3″. Both the preceding and following sides of the penumbra were of the same width, viz. 3″, though if there were a difference, the "following," or *inner*, side was the wider of the two, by perhaps 0″·5. There was at least a *sensible* difference (though exceedingly small, truly) in the opinion both of myself and a perfectly impartial fellow-observer, who, in fact, did not know what was expected one way or the other.

I regret that there should have been any divergence of opinion between Father Perry on the one side and Mr. Ranyard and myself on the other, though that divergence is evidently

extremely slight. What we require, however, in such investigations are actual measurements by micrometrical appliances.

Since writing the above I have carefully looked over the four last volumes of my drawings of solar spots, deposited in the library of our Society, extending over a period of ten years, from 1864 to 1873 inclusive, and find that during that period, whilst there are eleven instances fairly adverse to any phenomena of the foreshortening in question, there are seven instances fairly in its favour, but generally only to a slight degree.

With regard, however, to the very marked divergence between Dr. Wilson's observations of the last century and those of both Mr. Ranyard, Father Perry, and myself on this present occasion, we shall probably come to the conclusion that there are different classes of spots, and which, amongst other peculiarities, manifest very different degrees of profundity, as well as of general form, permanency, and even colour, as was observed in the month of April last by Mr. B. J. Hopkins, as well as on other occasions by Mr. Lockyer, myself, and, I doubt not, other observers.

Lastly, it may be as well to remember that the Sun's vastly deep and diaphanous atmosphere (whose existence is evinced by the very considerable diminution of the Sun's luminosity all round the margin of that luminary) must have some effect in modifying the appearances of all features situated on and near the limb. The effect of the refraction of the rays proceeding from the Sun's surface would be to cause somewhat more than half the disc to be visible to an observer on the Earth, and hence to retain solar features in sight which would otherwise have passed out of view; and also to flatten and level prominences standing on the limb, and possibly also to slightly increase the apparent width of the outer side of a penumbra somewhat more in proportion than the inner side, but from a different cause to that assigned to this by Dr. Wilson.

It was Professor Challis, I think, who many years ago drew attention to this modifying of objects on the Sun's limb, by virtue of the refraction exerted by the solar atmosphere.

The Fourth Satellite of Jupiter during Superior Conjunction on the Night of April 5, 1886. By Edmund J. Spitta.

A frequent explanation of the anomalous appearance which the satellites of *Jupiter* sometimes present during transit, but which is scarcely admitted to be adequate or satisfactory, is based upon the suggestion that the illumination of the hemisphere of the satellite then presented to the Earth is not equally distributed; and this explanation has been still further complicated by the suggestion that owing to the period of rotation possibly corresponding with that of revolution, this particular hemisphere is only seen at the time of transit, and that then the effect of the unequal illumination is aggravated by the contrast with the bright disk of *Jupiter*.

It seemed to me desirable, therefore, to call attention to the opportunity of examining the satellite at a time when the same hemisphere—neglecting libration—must necessarily be presented to the Earth as when the transit is in progress, and when the phenomenon would not be complicated by the background of *Jupiter's* disk, on which it is usually exhibited. Such opportunities of varying the condition under which the satellite can be seen occur but rarely, as a reference to the *Nautical Almanac* clearly shows, and I regret that the observations yet published have not been more numerous or definite. I present here those that have come under my own notice, or have been forwarded to me in consequence of the appeal I made to those interested in the subject.

All correspondents complain of the badness of the weather; hence my earliest note is at 8 P.M., when the satellite was to the north of its primary at its following limb. It was then bluish, and seemed scarcely brighter than the neighbouring portion of *Jupiter*. Mr. Denning thought it appeared as if veiled around its periphery and brighter at its centre. This was with his 10-inch Reflector; power, 252.

An hour later it was closer to *Jupiter*—almost touching it; it then appeared fainter than ever; and when compared with the nearest satellite, which was the first, it was found to reflect only one-fourth of the light of that satellite. (This observation was made, I am informed, with a photometer constructed after the plan suggested by Prof. Pritchard.) A friend and myself made our earliest observation at 10.30 P.M., and we both came to the conclusion that the size of the satellite appeared much reduced, and later still we thought its light was also diminished to about two-thirds of its usual quantity; but of course no definite allowance could be made for the effect of the adjacent luminary. I may add that I have witnessed a great number of transits of *Jupiter's* satellites, and since this observation have examined others, to note if the light of the other satellites when nearing transit was reduced as much as that of the fourth appeared to be at the time above mentioned, and I have come to the conclusion —however imperfect such eye-estimations may be—that they are not.

By a second photometric comparison between the light of the fourth with that of the first at 11.30 P.M. I am informed it was found that the reflected light had become increased, being then only one-third less than the comparison satellite; but whether this was due to the altered position of the satellite or to its reflective power having become greater, I am not in a position to state. Bad weather prevented any further observations either of the satellite or of its shadow.

Most observers reported fair definition. The instrument used here was the 10-inch Calver Reflector; power, 150.

Ivy House, Clapham Common,
1886, *April.*

Observations made at the U.S. Naval Observatory, Washington.
By Professor A. Hall, U.S.N.

(*Communicated by Commodore Geo. E. Belknap, U.S.N., Superintendent.*)

Conjunctions of the Satellites of Saturn.

Date.	Wash. M.T. h m	Pos.	Wt.	Remarks.
	Mimas.			
1886, Jan. 7	8 57·5	N	3	Excessively faint
Feb. 9	8 23·3	N	3	A little doubtful
	Enceladus.			
1885, Dec. 26	9 46·3	N	1	Very faint
1886, Feb. 23	7 34·3	N	3	
Mar. 10	9 15·5	N	3	
	Tethys.			
1885, Dec. 11	10 5·0	S	3	
12	8 53·6	N	3	
28	9 48·2	S	2	Cloudy
1886, Jan. 14	9 30·4	S	3	
15	7 58·0	N	3	
Feb. 1	7 38·5	N	3	Hazy
16	10 1·6	N	3	
17	8 44·4	S	3	Faint; hazy
18	7 26·7	N	—	
Mar. 5	9 43·9	N	2	
6	8 37·4	S	3	
7	7 7·9	N	3	
	Dione.			
1885, Dec. 11	9 37·0	S	3	
26	10 43·8	N	2	
1886, Jan. 6	9 16·4	N	2	
Feb. 1	9 5·9	S	3	
16	10 5·6	N	3	
Mar. 10	7 36·3	N	3	
14	10 19·8	S	2	
	Rhea.			
1885, Dec. 7	9 27·9	S	3	
16	10 12·8	S	4	
25	10 45·3	S	2	
1886, Feb. 15	8 23·5	N	2	Very unsteady
24	9 14·7	N	3	
Mar. 5	10 2·4	N	3	
14	10 59·0	N	2	

Satellites of Mars.

Deimos.

Date. 1886.	Wash. M.T. h m	P. °	Wt.	Wash. M.T. h m	S. ″	Wt.	Remarks.
Mar. 1	10 36·8	112·62	3	10 41·8	47·57	3	Windy; sat. faint
3	9 24·4	299·08	3	9 34·4	44·95	3	Very faint
4	12 12·4	283·05	3	12 20·9	47·75	3	Faint
4	12 31·4	283·58	3	12 38·4	48·19	3	
6	10 35·3	105 88	2	10 42·8	48·58	2	Very faint; sky misty

Comparison with Marth's Ephemeris.

C − O.

	Δp	Δs
Mar. 1	−1·02	+0·05
3	−1 07	−0·36
4	−1·47	−1·42
4	−0·49	−1·14
6	−0·05	−0·49

Companion of Sirius.

Date.	Sid. Time. h	P. °	S. ″	Wt.	Remarks.
1886·180	5·5	30·7	7·11	3	Faint
·183	5·5	32·7	7·56	2	Very faint
·194	5·7	32 4	7·11	3	Faint
·202	6·1	33·5	7·75	2	Very faint
·273	8·0	29 5	7·38	2	Faint; 15 inches aperture; double weight
·295	8·6	28·1	7·40	2	Faint; 15 inches aperture; double weight

Mean Result.

	P	S
1886·221	30°·56	7″·39

Remarks.—The conjunctions of the satellites were observed by placing the wire of the micrometer perpendicular to the major axis of the ring by means of the angle of position given in the *American Ephemeris* for 1886, p. 481. The conjunction of *Mimas* on January 7 was observed by Lieut. W. H. Allen, U.S.N., and that of *Tethys* on February 18 by Mr. G. Anderson.

The satellites of *Mars* were very faint, this opposition being the most unfavourable that will happen between 1877 and 1892. The outer satellite could be observed fairly well, but the inner one was seen only once, and then it was so faint that no measurement was made. It appeared to be near the predicted place. The weather was very unfavourable. The elements of these

satellites are probably nearly correct, but the approaching oppositions will furnish better observations for correcting their motions. The preceding observations have been corrected for differential refraction. The correction for the figure of the planet is not sensible.

The companion of *Sirius* was very faint, and was a difficult object to observe. When the images were blazing it was found that the companion was seen better with the aperture reduced to fifteen inches. The last two observations were made with the smaller aperture, and they have been given double weight.

1886, *May* 5.

Observations of Fabry's Comet (*d.* 1885). By L. A. Eddie.

The recent abnormally clouded condition of our atmosphere, which has been attended by floods of rain and dense fogs, has prevented earlier observations of this comet, which was discovered on December 1 last by M. Fabry at Paris in the northern heavens, and which reached its perihelion about April 6, soon after which passage it was to reveal itself to star gazers in the southern hemisphere. This visitor was seen here for the first time on May 1, and was observed by myself with optical aid on May 2 from 6.30 to comet-set, about 7.40, when the head sunk below the western horizon. When first seen on this occasion it was about 18° above the western horizon, and was a very conspicuous object, with a head of tolerable brilliancy about 15′ in diameter, and possessing a broad tail widening out to about 1½°, and traceable to about 9° from the nucleus. It shone with a pale straw-coloured light, and the tail appeared perfectly straight without any sign of curvature, but sharply defined. It was situated in the constellation *Eridanus*, about 3° N.W. of *ν Eridani*, just beneath the well-known constellation *Orion*. The tail stretched upwards to a point about half way between *β Eridani* and *η Orionis*. When viewed through the telescope it revealed a bright and well-condensed nucleus, surrounded by a broad but less luminous coma, and conveyed the impression of a brilliant pellet well sunk in the midst of less luminous gas-jets, emitted by itself from around its periphery, which, after extending towards the Sun for a certain distance, were bent back upon themselves and streamed in the opposite direction, gradually widening out, and thus forming an elongated and truncated cone of extremely rare gaseous matter, with a ball of more condensed matter in its longitudinal axis, and situated at a short distance from the narrow extremity. A very small star, probably about the eighth magnitude, was seen through the coma close behind the nucleus, and another again a little higher up in the tail. The nucleus was eccentrically placed within the coma, and the striæ or hair-like nature of this latter was very

marked. The theoretical brightness of the comet now was about 500 times greater than when first discovered by Fabry.

May 3: cloudy and wet. Comet visible for a short interval only, and then indistinctly through misty clouds; it had moved about 9° to the east, and was now placed nearly half way between β and ψ *Eridani*. Nucleus appeared elongated; but this may be owing to the mist.

May 4: evening clear; comet about 7° higher than previous evening, but fading rapidly; could be detected by the naked eye a little before six o'clock. Tail could not be traced this evening further than 6° from the head, but seemed slightly curved, with the convex and more sharply defined margin on the south. It is now situated in the constellation *Lepus*, close to a sixth-magnitude star, near the tip of the ears of the hare. The nucleus, when viewed through the $9\frac{1}{2}$-inch Calver, appeared very bright and condensed, and sharply defined, and of a ruddy brown colour, surrounded by a pale yellow coma of hyperbolic form, with its vertex almost pointed, and spreading out considerably in a lateral direction, so as to give the appearance of its possessing two luminous wings. The portion immediately behind the nucleus was comparatively dark, and the nucleus itself was more sharply defined on its eastern border; there was a considerable breadth of coma preceding the nucleus. The coma presented a granulated rather than a cometic appearance.

May 5: evening clear. Comet visible from 6 o'clock. Tail very faint, 5° in length. In $9\frac{1}{2}$-inch Calver, nucleus very bright and sharply defined. There was a slight trace of the coma preceding the nucleus, forming two or three cometic envelopes of parabolic form; the lateral streams, as seen on the previous evening, were very faint. It was situated near ζ *Leporis*.

May 6: sky hazy, with thin cirri clouds. Comet very dim.; tail scarcely perceptible, about 4° in length. Nucleus in telescope dim, lateral streams not detectable. It was situated in *Lepus*, and a straight line drawn from α, through β *Canis Majoris*, and continued as far beyond β as this star is from α (*Sirius*), would mark almost the exact spot that the comet occupied at 7 o'clock.

May 7: sky hazy. Comet very faint. Tail about 2° in length. Not observable by telescope.

May 8: evening clear. Comet faintly visible, but tail scarcely discernible, being eclipsed by moonlight.

May 9 and May 10, clouded.

May 11: sky clear for short interval, moon very bright. Comet just discernible to the naked eye, situated in *Canis Major*, forming a right angle with ϵ and 22 *Canis*, about twice as far from ϵ as this star is from 22 *Canis*. When viewed through a $3\frac{1}{2}$-inch Refractor appeared like a nebula slightly condensed in centre, and of ragged outline. In the $9\frac{1}{2}$-inch Calver the nucleus shone brightly, and still retained the deep ruddy colour which is a distinguishing feature in this comet. This is the

first comet that I have observed whose nucleus has been of so deep a tint; the outlying coma is still a pale yellow.

May 12: comet very faint, only just distinguishable to unaided vision as a dim misty spot. It is now becoming a very insignificant object in the telescope, the background being so brilliantly illuminated by moonlight.

May 13: not observed.

May 14: comet invisible to naked eye. Moon very bright, easily found with finder. Has just entered the constellation *Argo Navis*. Nucleus still well condensed. No detectable change. Very mean object now.

Approximate Right Ascension and Declination at 7.30 P.M.

Date.	R.A.	South Dec.	Date.	R.A.	South Dec.
	h m	° ′		h m	° ′
May 2	4 22	1 30	May 7	6 15	22 0
3	5 12	6 0	8	6 28	24 30
4	5 20	12 0	11	7 4	29 30
5	5 37	15 0	12	7 15	30 15
6	5 54	19 15	14	7 24	33 30

Cape of Good Hope, Graham's Town.

Sextant Observations of Fabry's Comet.

(*Communicated by Captain H. Toynbee.*)

Observations made on board the barque "Viola," by Captain J. H. Price.

1886, May 2, 7.20 P.M. ($10^h\ 5^m\ 4^s$ G.M.T.) Lat. 16° 24′ N.; Long. 35° 31′ W. Comet's position was as follows:

	° ′
Nucleus to *Betelgeuse*	20 5
„ „ *Rigel*	10 20

Tail about 5° in length.

Observations made on board the H.M.S. "Spartan," by Captain A. McLean Wait.

1886, May 4, 8 P.M. ($8^h\ 45^m\ 55^s$ G.M.T.) Lat. 1° 54′ S.; Long. 12° 20′ W. Barometer 30·138. Thermometer, wet bulb 78°, dry bulb 85°. Comet's position was as follows:—

	° ′
Nucleus to *Betelgeuse*	20 35
„ „ *Sirius*	19 30

Altitude of nucleus 11°. Tail about 2° long, like a partially closed fan. Brightness about equal to a second-magnitude star.

Observations made on board the ship "Melbourne," by Captain C. F. Cooke.

1886, May 2, $9^h\ 11^m$ G.M.T., $7^h\ 30^m$ A.T.S.; D.R. Lat. 1° 20′ S.; D.R. Long. 25° 20′ W.

	°	′	″
Distance of Comet from *Betelgeuse* ...	20	9	
„ „ „ *Sirius*	34	1	30
Altitude	4	30	

May 3, $8^h\ 50^m$ G.M.T., $7^h\ 4^m$ A.T.S.; D.R. Lat. 0° 43′ S.; D.R. Long. 26° 36′ W.

	°	′
Distance of Comet from *Betelgeuse*	18	58
„ „ „ *Sirius*	26	28

May 5, $9^h\ 26^m$ G.M.T., $7^h\ 30^m$ A.T.S.; D.R. Lat. 1° 14′ N.; D.R. Long. 27° 20′ W.

	°	′
Distance of Comet from *Betelgeuse*	23	36
„ „ „ *Sirius*	14	0

May 7, $9^h\ 8^m$ G.M.T., $7^h\ 20^m$ A.T.S.; D.R. Lat. 3° 44′ N.; D.R. Long. $28^h\ 22'$ W. Comet very faint. Observed through ordinary telescope; tail very short, large illuminated space round nucleus, which has always appeared larger than any comet I have ever seen before.

	°	′
Observed angles with *Sirius*		26
„ „ „ *Procyon*	33	50
„ „ „ *Canopus*	29	18

Observations of Comet a 1886 (*Brooks*), *made at the Royal Observatory, Greenwich.*

(*Communicated by the Astronomer Royal.*)

The observations were made with the East or Sheepshanks Equatorial, aperture 6·7 inches, by taking transits over two cross-wires at right angles to each other, and each inclined 45° to the parallel of Declination.

Comet a 1886.

Greenwich Mean Solar Time.	Observer.	☄ − ✶ R.A.	Corr. for Par. and Refract. in R.A.	☄ − ✶ N.P.D.	Corr. for Par. and Refract. in N.P.D.	No. of Comp.	Apparent R.A.	Apparent N.P.D.	Comp. Star.
1886. h m s		m s	s	′ ″	″		h m s	° ′ ″	
May 20 10 3 26	H.	−0 8·37	+0·21	−12 6·9	−18·8	3	2 43 53·30	45 12 29·2	*a*
10 26 20		+0 36·37	+0·08	− 4 59·9	−12·5	3	2 44 5·62	45 13 6·7	*b*
10 40 17		+2 22·30	+0·02	+ 0 53·9	− 6·1	2	2 44 8·58	45 13 49·9	*c*

Mean Places of Comparison Stars.

Star's Name.	R.A. 1886·0.	N.P.D. 1886·0.	Authority.
	h m s	° ′ ″	
(*a*) W.B. (2) II. 1016	2 44 1·74	45 24 44·9	Weisse's Bessel (2)
(*b*) Lalande 5203	2 43 29·45	45 18 9·2	Lalande
(*c*) W.B. (2) II. 961	2 41 46·54	45 12 52·0	Weisse's Bessel (2)

The observations are corrected for parallax and refraction. The initial H. is that of Mr. Hollis.

Royal Observatory, Greenwich:
1886, *June* 9.

Note with respect to the Invention of the Achromatic Telescope.
By A. C. Ranyard.

At a recent meeting of the Astronomical Society I mentioned that there was no account to be found in the Law Reports of the time of the action brought in 1766 by Peter Dollond, the son of John Dollond, against Champness, an optician in Cornhill, for infringing John Dollond's patent for making achromatic telescopes. The importance of any contemporary reference to this action arises from the fact that Champness, in his defence, proved to the satisfaction of Lord Mansfield that achromatic telescopes had been made by Mr. Moor Hall in 1733, that is, twenty-five years previously to the date of John Dollond's patent. It is known* that evidence was given by workmen who had been employed by Mr. Moor Hall to make achromatic lenses for him, but it is not known whether the workmen actually came into Court and were cross-examined, or whether their evidence was given by affidavit.

Mr. C. T. Martin saw a report of the discussion at the Astronomical Society's meeting, and wrote to suggest to me that I should search the Common Law Judgment Rolls preserved in the Record Office. I was under the impression that the action had been tried in the Court of King's Bench; for we know from subsequent references to the case that it was tried before Lord Mansfield, who was Chief Justice of the King's Bench, from the date of his appointment in November 1756 on till 1788. There is in the library of the Record Office a complete index of the actions tried during this period in the Courts of King's Bench and Common Pleas. I therefore searched the index of cases in the King's Bench for the names Dollond and Champness, but with no success, and I was about to give up the search, when the librarian suggested to me that Lord Mansfield might have sat to relieve a judge in another Court, and he suggested that I should search the index of cases in the Court of Common Pleas. There I found in the list of actions tried in the Easter Term of the seventh year of George III. the entry:—

> *James Chumpneys*, late of London, optician, to answer Peter Dollond, administrator of John Dollond. Damages £204. H. 6 Geo. III. 626 Middx.

The latter figures refer to the record of Judgments. On sending for the Judgment Roll for the Easter Term of 7th Geo. III. I found that Judgments numbered 622 to 627 were missing, and on searching through the file of affidavits made in the Common Pleas in Easter Term of 7th Geo. III. I was

* For references with respect to this evidence see *Astronomical Register* for 1881, pp. 196, 197.

unable to find any referring to the matter. The next day, on turning to the short notice of the decision given in the *Gentleman's Magazine* for 1766, p. 102, I found that judgment was given on Tuesday, February 18, 1766, which would correspond to the 6th year Geo. III., and I concluded that I must have made a mistake of a year in the index reference in the Record Office library. On sending for the Judgment Roll for Hilary Term of 6th Geo. III. I found that, by a curious coincidence, Judgments 622 to 627 were again missing. There seems to be no connection between the two Rolls. The librarian informed me that at that period registration of judgments was not compulsory, and as a small fee was charged for registration, several of the judgments—perhaps one in five—are missing.

By the courtesy of Mr. Selby, the Chief Librarian, I was allowed to have all the Common Pleas documents, consisting of *posteas* and *affidavits* for the years 1765, 1766, and 1767, in a private room to search at my leisure. The search occupied some days, as there were nine sacks full of *posteas*, and I went through all the affidavits without trusting to the rough alphabetical arrangement in which they have been stored. But I found nothing referring to the case of Dollond *v.* Champness. I was, however, rewarded by finding affidavits in two other actions brought by Peter Dollond against opticians to recover damages for infringing his father's patent. These actions seem to have been brought subsequently to the test case against Champness, or Chumpneys, and to have been settled as a matter of course. The affidavits only refer to formal matters connected with the service of writs, but they serve to show that at least three opticians besides James Champness had acted on the supposition that Dollond's patent would not be upheld.

Before judgment was given in Dollond *v.* Champness, Peter Dollond had commenced an action against Francis Watkins and Addison Smith as joint defendants. They are described as both of St. Martin's Lane, opticians. Subsequently Peter Dollond commenced an action against Henry Pyefinch, who is described as an optician of Cornhill. The affidavits in the first of these actions will be found filed under 'Dollond' in Easter Term of 6th Geo. III., and an affidavit in the second action under 'Dollond' in Michaelmas Term of 8th Geo. III. In both actions Messrs. Edmonson and Potts, of Bury Street, acted as attorneys for Peter Dollond. They were therefore probably the attorneys for Dollond in his action against Champness. The fact is worth noting, as papers connected with the case may have been preserved by the firm. A member of Mr. Dollond's family informed me some years ago that he had made inquiries, and believed that no papers referring to the case had been preserved by the Dollonds.

On the Superiority of Zinc and Steel Pendulums.

By Thomas Buckney.

At the last meeting of the Royal Astronomical Society, Dr. Leonard Waldo, of Yale College, U.S.A., was good enough to give us an account of certain improvements in astronomical clocks, or, as he preferred to call them, "clocks of precision." The net result of these improvements was stated to be a reduction of about 25 per cent. in the average errors of clock rates, and this as compared with the best previously existing clocks. This, if substantiated, would be a most important gain; but, unfortunately, Dr. Waldo gave us no figures in support of his statement, which, therefore, goes for nothing more than a mere expression of his own individual opinion. Now I cannot help thinking that when a statement of this kind is made to such a Society as this, it ought, to have any value, to be accompanied by the data on which it is founded, so that anyone interested in the matter may be able to weigh the facts and verify the deductions made from them. In the absence of such data we can only discuss the methods adopted, and endeavour to satisfy our own minds that they would probably produce the result said to be attained.

Now, the Yale clock or clocks which appear to have been constructed under Dr. Waldo's direction embody the following features, and it is to them that he attributes their excellent performance. They have:

1st. A heavy mercurial pendulum, the mercury weighing from 45 to 60 lbs. The jars containing the mercury and the pendulum rods are of steel.

2nd. The suspension of the pendulum is on a knife-edge, in lieu of the spring commonly used.

3rd. The escapement is a modification which is stated to be an improvement of one of the forms of Denison's Gravity Escapement.

4th. The support of the pendulum is a massive iron arch spanning two piers of masonry similar to the piers of a transit instrument, and it is between these piers that the pendulum swings. The clock movement is supported on a bracket placed between the piers, around which some arrangement of steam pipes is introduced for the purpose of regulating the temperature.

Well, now, the first thing that crosses one's mind is that some of these improvements are old acquaintances. Let us see, therefore, what is known about them, taking each one in the order already given. In order not to weary you I will, if possible, confine my references to one little book, which, although it has run through several editions and was written by a gentleman who is a Fellow of this Society, Dr. Waldo must, I think, imagine that few of us have seen. Perhaps Dr. Waldo himself

has not seen it, in which case I commend it to his notice, as the author, who is President of the British Horological Institute, is generally recognised as an authority on these matters. It is entitled "A Rudimentary Treatise on Clocks, Watches, and Bells," and I refer to it to prove that heavy mercurial pendulums, such as Dr. Waldo recommends, have been in use for many years. For in an edition of the work published in 1874 the author says, "In 1863 I had to calculate the height (that is, of the mercury) for the iron jar of a pendulum intended to weigh 40 lbs." Besides, there is on the Water Clock used for driving the Greenwich Equatorial, a mercurial pendulum of about this weight, which was made by my own predecessors probably thirty years ago. The jars of these pendulums were of iron it is true, but this is a point which is absolutely immaterial.

Now as to the knife-edge suspension. In a still older edition of this same rudimentary treatise—one published in 1850—I find it stated, "Occasional mention may be seen in books of pendulums vibrating for several hours on knife-edges—that is, on a suspension much like that of a scale beam. Such a suspension is the best for experiments to ascertain the undisturbed time of vibration due to gravity only, excluding the elasticity of a spring; but it is not to be inferred that it will answer in practice for a permanent pendulum of the proper weight for a clock, for even if the knife-edges and the planes on which they stand are made of the hardest stones, it is found that they soon suffer from the severe pressure, and introduce an amount of friction which is fatal to the accuracy of the pendulum." Well, since this condemnation, as far as I am aware, as regards clocks, the knife-edge suspension has remained dead and buried until Dr. Waldo dug it up and brought it as treasure-trove to our last meeting.

The modification made at Yale in the gravity escapement may or may not be an improvement. In either case it can have very little effect on the time-keeping of the clock, as it affects only the lifting of the arms by the clock train. As far as the pendulum, which is the time-measurer, is concerned, the escapement remains precisely what it was before the alteration.

I quite agree with the importance Dr. Waldo assigns to a rigid support for the pendulum. And here we are perfectly in accord with the author of the "Rudimentary Treatise," who points out its advantages in every edition of his work. There is a simpler and probably a better way of obtaining the desired rigidity—that is, by bolting a massive iron bracket to one of the main walls of the building in which the clock is to be placed, and hanging the pendulum from that; but it is obviously immaterial in which way the desired rigidity is obtained so that it be really secured. We now come to the last of Dr. Waldo's improvements —the introduction of steam piping round the piers on which his clock is fixed; and this certainly is a novel proposal. I will not, however, say any more about it, as I hope to be able to show that it is, or ought to be, quite unnecessary. But my purpose

in the present paper is not so much to show the antiquity of the novelties brought before us by Dr. Waldo as to show that they are not well calculated to yield the result he claims for them; and this more particularly applies to the pendulum, to which, as the time-measurer, I propose to confine myself. As it is possible that in what I am about to say I may repeat something I said at the last meeting, I must ask you to excuse me. Having had no idea that this subject was to be brought forward, I had no opportunity of referring to my memoranda bearing on it, and so was placed at a great disadvantage, as I had to speak entirely from memory. But the matter seems to me of sufficient importance to induce me to trespass on your time this evening, as I should be sorry to find Dr. Waldo's paper accepted as an accurate description of our knowledge as regards astronomical clocks.

Well, about eighteen years ago, it fell to my lot to make designs for four astronomical clocks, one of which was destined to become the sidereal standard clock of the Greenwich Observatory; the others were to be used as transit clocks in the British Transit of *Venus* Expeditions. The first of these clocks was by far the most important, and was to be constructed so as to embody the features of a specification drawn up by Sir George Airy. But the only point we have to consider here is the pendulum; which was to be a mercurial one. The other clocks were fitted with zinc and steel compensation pendulums, constructed in what was then the ordinary way; that is, the bobs, which were cylinders of lead, were supported at one end, so that the expansion or contraction of the bob itself entered into the compensation. These four clocks, completed at about the same time, were set going in a quiet room, in which there was a gas-stove, by which the temperature could be raised without difficulty to 90°. As set going under these circumstances, these clocks had a rather considerable temperature error; but what seemed at first rather curious was that this did not at first show itself in a very marked way, but kept on increasing for two or three days until the clocks finally settled down to a steady rate due to the higher temperature, and a similar sluggish action was noticed when the temperature was allowed to run down. It became evident that this sluggish action was due to the more slender parts of the pendulum (the rods and spring) taking up the new temperature very quickly, whilst the bobs, being much more bulky, required a much longer time to do this. I would here mention that, fully recognising the advantage of a heavy pendulum, I had given the Greenwich clock a jar $2\frac{3}{4}$ inches internal diameter, holding about 20 lbs. of mercury, the diameter of the jars in ordinary use being 2 inches, which would hold about 12 lbs. of mercury. Considering the matter, it occurred to me that if we were to eliminate from the compensative action of the zinc and steel pendulums the expansion of the bob, which we could easily do by suspending it at its centre of gravity, and rely entirely on the zinc and steel tubes and rod, matters would

be very much improved, as then the compensation would be effected by parts having pretty much the same bulk, and therefore likely to act simultaneously. And so it turned out. These zinc and steel pendulums were reconstructed on the principle above indicated, and the clock rates became very soon all that could be wished. But the Greenwich clock could not be altered in this way, as in a mercurial pendulum the compensation is and must be effected entirely by the expansion of the bob, and there seemed no satisfactory way of increasing the rapidity of its action whilst maintaining its weight. In the end I determined to lay the matter before Sir George Airy, and request his sanction to the removal of the mercurial pendulum from the Greenwich clock, and the substitution in its place of a zinc and steel pendulum on the new principle. The change produced the same rapid and simultaneous action obtained in the other clocks, and has, I venture to think, been quite satisfactory. It was in this way that it came about that the Greenwich clock has a zinc and steel pendulum, instead of the mercurial one originally intended for it.

Some time after this, being desirous of investigating more fully this sluggish action of the mercurial pendulum, I made the two following experiments, which are rather interesting. I took an ordinary mercurial pendulum jar of two inches diameter, and, having removed the steel rod, I cemented in its place a glass tube and filled the top completely with mercury, and also the tube to about half of its height. I marked the height of the mercury in the tube, and noted the temperature, which was about 60°. I then plunged the pendulum into hot water—temperature about 150°—and the result was as follows:

Almost immediately the mercury sunk in the tube considerably, but in less than a minute it began to expand with the heat, and then rose rapidly. The first effect of a sudden rise in the temperature was, however, to make the mercury fall—a phenomenon due evidently to the expansion of the jar by the sudden rise of temperature taking place before this new temperature was felt by the mercury to a sufficient extent to counteract it. This experiment, which was repeated several times with the same result, shows that the first effect of a change of temperature on a mercurial pendulum is in the inverse direction, and aggravates the error it is intended to correct. This goes on, although gradually lessening, until the mercury acquires the new temperature. But before this happens, an error has arisen which cannot be corrected; it may be, however, and probably is in many instances, masked by over-compensation. In any case irregularities of rate are engendered.

The next experiment was made as follows:

Two mercurial pendulum jars were taken, one of 2 inches diameter, and the other $2\frac{3}{4}$ inches diameter, the latter being the one discarded from the Greenwich clock. These were filled with mercury to about the height that would be required to com-

pensate the pendulums. In the place of the pendulum rods thermometers were inserted, the bulbs of which were plunged into the centre of the mercury and so arranged as to project above the caps of the jars to such a height as to allow the temperature to be read off. Similarly, a short length of the outer steel tube of a zinc and steel pendulum was taken, the lower end being stopped by a cork. Inside this was placed a short length of the zinc tube, and inside this, in lieu of the steel rod, was a thermometer as in the mercurial jars. A large saucepan was then obtained from the kitchen, and having adjusted the bobs and compensating piece of zinc and steel on a piece of non-conducting material in the bottom of the saucepan, water was put in until it nearly reached the top of the bobs. The saucepan was then placed on a gas-stove and the temperature raised to about 200°, and the whole kept in this state for a couple of hours, by which time the high temperature had thoroughly pervaded the masses of metal, as was shown by the readings of the thermometers. The bobs were then removed from the water and placed at a certain distance apart in a quiet room, the temperature of which was about 60°. Now, the object of this experiment was to ascertain how long each of these pendulums took to adapt itself to a new temperature; and the three thermometers were therefore watched, and the time each one took to cool down a certain number of degrees was noted.

Well, then, to cool down from 180° to 140° the zinc and steel pendulum took 4 minutes; the 2-inch mercurial one 21 minutes, and the $2\frac{3}{4}$-inch mercurial $29\frac{1}{2}$ minutes. To cool from 140° to 120° the times occupied were respectively 5, 15, and $22\frac{1}{2}$ minutes; from 120° to 100°, 8, 24, and 38 minutes; from 100° to 90°, 5, 19, and 29 minutes; from 90° to 80°, $8\frac{1}{2}$, 28, and 49 minutes; and from 80° to 70°, below which point the observations were not carried, the times were $26\frac{1}{2}$, 70, and 115 minutes. And the times respectively over the whole range of the experiment—that is, from 180° to 70°—were 57, 177, and 283 minutes.

The importance of these results will be more apparent and better appreciated if arranged in a table, as follows:

Temperature.	180° to 140°	140° to 120°	120° to 100°	100° to 90°	90° to 80°	80° to 70°	180° to 70°	Ratios
Zinc and Steel	4	5	8	5	$8\frac{1}{2}$	$26\frac{1}{2}$	37	1
2-inch Mercurial	21	15	24	19	28	70	177	3
$2\frac{3}{4}$-inch Mercurial	$29\frac{1}{2}$	$22\frac{1}{2}$	38	29	49	115	283	5

Now, this experiment, to my mind, shows clearly that in any given change of temperature a zinc and steel pendulum in the compensating action of which the bob has no share is three times as rapid as a mercurial pendulum with a 2-inch jar, and five times as rapid as one with a $2\frac{3}{4}$-inch jar, and the clock errors would be in the proportion indicated by these figures. Indeed, it is certain that the zinc and steel pendulum would in

practice show even better than it did here; because in these pendulums the steel tube is cut through so as to expose nearly the whole of the zinc tube, and this latter is pierced with many holes to allow the temperature to reach the inner rod, whereas in this experiment the tubes were entire. As regards the two mercurial pendulums, it will be seen, as might have been suspected, that the larger one took longer to cool than the smaller, although possibly one would not have thought that the difference would be so great. It seems to vary rather more quickly than the area, but this may be due to some difference in the thickness of the walls of the two jars. Let us, therefore, assume that the sluggishness varies as the area, and see how the large bobs containing 45 and 60 lbs. of mercury, recommended by Dr. Waldo, would come out. These would have an internal diameter of four and five inches approximately, and the areas would be as follows:

Diameter	2	$2\frac{3}{4}$	4	5
Areas	3·14	5·93	12·56	19·63

If, therefore, my experiment holds good for these larger diameters (and I do not see why it should not do so), Dr. Waldo would find that his 4-inch pendulum would take $12\frac{1}{2}$ times as long as the zinc and steel one, and his 5-inch pendulum 20 times as long, to adapt themselves to a new temperature, and as compared with the 2-inch pendulum in common use they would be 4 and $6\frac{1}{2}$ times as long. And it will be evident that the more the size of the jar is increased the greater the error will be. In fact, of mercurial pendulums it may be said the bigger they are the worse they are. The errors of Dr. Waldo's clocks, due to the sluggish action of his large pendulums, may, no doubt, be lessened by a careful manipulation of his steam pipes; but clocks cannot always be treated as exotics: we want something more hardy. I would urge most strenuously the importance of protecting clocks from sudden and great changes of temperature. It is, indeed, absolutely necessary that this should be done, and if a dry underground chamber can be found and a clock set up in such a place, where only gradual variations of temperature occur, one may reasonably look for a good performance. But such protection cannot always be found, and in order to prove that a zinc and steel pendulum can, and does, give a good result under very trying circumstances, I cannot do better than append the following rates of one of the three clocks referred to in the earlier part of this paper. These rates were taken for a period of six months, from September 3, 1871, to March 21, 1872, during which time the clock was going in one of the little huts prepared for the Transit of *Venus* Expedition, erected in the grounds of the Royal Observatory, and are extracted from the official records of that Institution.

Royal Observatory, Greenwich.

Errors and Rates of the Clock "Dent" 1914, on Greenwich Sidereal Time.

Date.	Clock slow.	Mean daily losing rate during each interval.	Average temperature of external air.	Extremes of temperature of external air.
	d h m s	s	°	° °
1871 Sept.	3 21·7 14 31·8	4·45	61	79–47
	17 21·5 15 34·1	4·02	52	68–39
	24 21·8 16 2·3	4·54	51	66–42
Oct.	1 22·6 16 34·2	4·43	51	65–40
	8 21·9 17 5·1	4·55	46	60–31
	15 21·6 17 36·9	4·47	54	68–41
	22 21·6 18 8·2	4·24	47	59–33
	29 21·3 18 37·8	4·25	46	55–35
Nov.	5 22·7 19 7·8	4·07	38	50–26
	12 22·2 19 36·2	4·24	35	51–20
	19 21·7 20 5·8	4·36	36	44–28
	26 21·7 20 36·3	4·33	36	43–28
Dec.	3 22·2 21 6·7	3·89	30	40–19
	10 21·9 21 33·9	4·67	41	47–27
	17 21·8 22 6·6	4·77	42	49–35
	26 0·1 22 45·2	4·74	43	48–33
	31 21·9 23 13·2	4·72	42	51–34
1872 Jan.	7 22·3 23 46·3	4·93	40	51–32
	14 21·6 24 20·7	4·79	39	48–28
	21 21·6 24 54·2	5·13	42	51–36
	28 22·1 25 30·2	5·17	45	54–37
Feb.	4 22·1 26 6·4	5·00	47	58–33
	11 22·2 26 41·4	4·96	44	54–37
	18 21·7 27 16·0	4·85	45	56–35
	25 21·8 27 50·0	4·88	46	58–32
Mar.	3 21·6 28 24·1	4·86	47	61–28
	10 21·6 28 58·1	4·73	46	59–28
	17 21·4 29 31·2			

During the whole time of rating, the clock was situated in one of the small Observatories intended for use in the observations of the Transit of *Venus*, 1874. No record of the temperature of the hut was kept, but its temperature would be nearly the same as that of the external air.

But our experience of these zinc and steel pendulums is not limited to these four clocks. There are three in use in the Royal Observatory, Greenwich, and others fitted with these pendulums will be found at the Observatories of Brussels, Liège, Copenhagen, San Fernando, Coimbra, Lisbon, Tokio, Hongkong, Cape of Good Hope, Port Elizabeth, and at many others; but these will suffice to show that they really do work well. A further improvement has recently been introduced, which allows an alteration in the compensation to be made by the astronomer without the slightest difficulty, and this without making a very wide difference in the clocks' rate; thus removing the possible objection that such an alteration could only be made by a clockmaker. I hope to bring this before you at some future time. As it is, I fear I have already detained you much too long. But I wished to show you that Dr. Waldo's novelties are not new, and that they are not likely to realise his expectations. I wished to point out the defects of the large mercurial pendulums he recommends; and I hope I have made it clear that, in a variable temperature, these pendulums do not and cannot act with the rapidity that is necessary. In order to maintain a steady rate, it is essential that the length of the pendulum—that is, the distance between the point of suspension and the centre of oscillation—shall, under all circumstances, be unaltered; a result which cannot be obtained with a pendulum, one part of which is affected by a change of temperature before the other part. If, then, my experiments are to be relied on, the conclusion must be drawn that the zinc and steel pendulum much more nearly fulfils the condition of simultaneous action, and is for this reason to be preferred. And this conclusion is justified by actual experience.

Ephemerides of the Satellites of Saturn, 1886–87. By A. Marth.

The five inner satellites deviate so little from the plane of the ring that their deviations are most suitably treated as latitudes above this plane, the ascending node N and inclination I of which, in reference to the plane of the Earth's equator, are here assumed to be

for 1887·0 $\quad N = 126°·5896 \quad I = 6°·9973$

The assumed longitudes of the five satellites in their orbits (i.e. their longitudes from the ascending node added to the Right Ascension N of the ascending node, reckoned from the point of the true equinox), referred to the time when the light arrives at the distance [0·950], are the following:

Greenwich Noon.	Mimas.	Enceladus.	Tethys.	Dione.	Rhea.
	°	°	°	°	°
1886, Oct. 10	0·183	180·228	158·187	272·596	35·843
Nov. 9	299·954	142·184	119·132	258·647	266·548
Dec. 9	239·726	104·141	80·076	244·699	137·252
1887, Jan. 8	179·499	66·098	41·021	230·751	7·957
Feb. 7	119·273	28·055	1·966	216·803	238·662
Mar. 9	59·048	350·011	322·910	202·854	109·366
Apr. 8	358·824	311·968	283·854	188·906	340·070
May 8	298·601	273·924	244·798	174·957	210·774

The adopted daily sidereal motions of the five satellites and the corresponding periods of their sidereal revolutions are:

	°	d h m s
Mimas	381·99243	0 22 37 5·70
Enceladus ...	262·73186	1 8 53 6·86
Tethys	190·69812	1 21 18 25·96
Dione	131·53503	2 17 41 9·33
Rhea	79·69012	4 12 25 11·87

The values for *Mimas* belong to 1887, Jan. 8, and depend on the hypothesis of accelerated motion.

In the following tables P denotes the position-angle of the minor axis of the ring, L+180° the planetocentric longitude of the Earth referred to the plane of the ring, Λ+180° that of the Sun, or Λ−L the difference between the two. The apparent equatorial diameter of the ball and the diameter of the outer rim of the ring depend still on Bessel's determinations. The assumed proportion of the polar axis of the ball to the equatorial diameter is 0·900.

In the tables for the satellites a and b are the semi-axes of the apparent orbits, l−L the longitudes of the satellites in their orbits, reckoned from the points which are in superior conjunction with the planet's centre or in opposition to the Earth in longitude. By adding to l−L the value of L from the first table, the longitudes l are found.

The values of P, a, b, and l−L are to be interpolated directly for the times for which the apparent positions of the satellites are required, the equation of light being already taken into account in the given values, and the rectangular co-ordinates x and y, reckoned parallel to the axes of the ring and expressed in seconds of arc, or, if polar co-ordinates are wanted, the position-angles p and distances s of the satellites in reference to the centre of the planet are then found by means of the equations

$$s \sin (p-\mathrm{P}) = x = a \sin (l-\mathrm{L})$$
$$s \cos (p-\mathrm{P}) = y = b \cos (l-\mathrm{L}).$$

Greeenwich Noon.	P	L	Latitude of Earth above plane of ring.	Latitude of Sun above plane of ring.	A−L.
1886.					
Sept. 30	352°·608	115°·925	−22°·895	−24°·491	−6°·502
Oct. 5	·602	116·217	22·819	24·455	6·594
10	·598	116·472	22·755	24·418	6·648
15	·595	116·684	22·703	24·381	6·659
20	·592	116·849	22·663	24·344	6·624
25	·590	116·966	22·636	24·306	6·541
30	·589	117·035	22·623	24·268	6·409
Nov. 4	352·588	117·055	−22·624	−24·230	−6·229
9	·588	117·026	22·639	24·192	6·000
14	·589	116·949	22·667	24·154	5·723
19	·591	116·825	22·708	24·115	5·400
24	·593	116·655	22·762	24·076	5·030
29	·596	116·440	22·829	24·036	4·615
Dec. 4	352·600	116·183	−22·907	−23·997	−4·159
9	·605	115·887	22·996	23·956	3·664
14	·610	115·557	23·092	23·917	3·135
19	·616	115·196	23·196	23·877	2·576
24	·623	114·810	23·306	23·836	1·991
29	·631	114·403	23·421	23·795	1·385
1887.					
Jan. 3	352·640	113·981	−23·539	−23·754	−0·765
8	·649	113·552	23·658	23·713	−0·138
13	·659	113·121	23·776	23·671	+0·491
18	·669	112·694	23·891	23·629	1·117
23	·679	112·277	24·003	23·587	1·732
28	·689	111·876	24·110	23·545	2·330
Feb. 2	352·699	111·497	−24·210	−23·502	+2·906
7	·709	111·147	24·303	23·459	3·453
12	·718	110·829	24·387	23·416	3·968
17	·726	110·548	24·462	23·373	4·446
22	·733	110·308	24·527	23·329	4·883
27	·739	110·111	24·582	23·285	5·277
Mar. 4	352·744	109·960	−24·626	−23·241	+5·624
9	·747	109·857	24·658	23·197	5·923
14	·748	109·803	24·679	23·153	6·173
19	·747	109·799	24·689	23·108	6·374
24	·745	109·844	24·688	23·063	6·525
29	·742	109·938	24·676	23·018	6·626

Greenwich Noon.	P	L	Latitude of Earth above plane of ring.	Latitude of Sun above plane of ring.	A−L.
1887.	°	°	°	°	°
Apr. 3	352·737	110·081	−24·653	−22·972	+6·678
8	·731	110·271	24·618	22·926	6·684
13	·723	110·507	24·572	22·880	6·644
18	·714	110·786	24·516	22·834	6·560
23	·704	111·107	24·449	22·788	6·434
28	·693	111·469	24·372	22·741	6·267
May 3	352·682	111·868	−24·285	−22·694	+6·063
8	·670	112·301	24·188	22·647	5·824
13	·658	112·767	24·081	22·600	5·552
18	352·645	113·264	−23·965	−22·552	+5·250

Greenwich Noon.	Diameter of Ball. Equat.	Phase prec. *l*.	Polar	Axis of Ring. Major	Minor	*Mimas.* a_1	b_1	l_1−L.	Diff.
1886.	″	″	″	″	″	″	″	°	°
Sept. 30	17·63	0·057	16·14	40·64	15·81	27·76	−10·80	23·63	1909·85
Oct. 5	17·78	·059	16·29	41·00	15·90	28·01	10·86	133·48	·89
10	17·94	·060	16·43	41·37	16·00	28·26	10·93	243·37	·93
15	18·11	·061	16·58	41·75	16·11	28·52	11·01	353·30	1909·98
20	18·28	·061	16·73	42·14	16·23	28·79	11·09	103·28	1910·02
25	18·45	·060	16·89	42·53	16·37	29·05	11·18	213·30	·07
30	18·62	·058	17·05	42·92	16·51	29·32	11·28	323·37	·12
Nov. 4	18·79	·055	17·20	43·31	16·66	29·59	−11·38	73·49	1910·16
9	18·96	·052	17·35	43·70	16·82	29·86	11·49	183·65	·20
14	19·12	·048	17·50	44·08	16·99	30·11	11·60	293·85	·24
19	19·28	·043	17·65	44·44	17·16	30·36	11·72	44·09	·28
24	19·43	·038	17·79	44·79	17·33	30·60	11·84	154·37	·31
29	19·57	·032	17·92	45·12	17·50	30·83	11·96	264·68	·34
Dec. 4	19·70	·026	18·04	45·42	17·68	31·03	−12·08	15·02	1910·37
9	19·82	·020	18·15	45·69	17·85	31·22	12·20	125·39	·39
14	19·92	·015	18·25	45·93	18·01	31·38	12·31	235·78	·40
19	20·01	·010	18·34	46·13	18·17	31·52	12·42	346·18	·41
24	20·08	·006	18·41	46·30	18·32	31·63	12·52	96·59	·41
29	20·14	·003	18·46	46·42	18·45	31·72	12·61	207·00	·41
1887.									
Jan. 3	20·17	0·001	18·49	46·50	18·57	31·77	−12·69	317·41	1910·41
8	20·18	follow.	18·50	46·53	18·67	31·79	12·76	67·82	·39
13	20·18	limb.	18·50	46·51	18·75	31·78	12·81	178·21	·36
18	20·15	0·002	18·48	46·45	18·81	31·74	12·85	288·57	·34
23	20·11	·005	18·44	46·35	18·85	31·67	12·88	38·91	·31
28	20·04	·008	18·39	46·20	18·87	31·57	12·89	149·22	·27

Greenwich Noon.	Diameter of Ball. Equat.	Phase foll. l.	Polar	Axis of Ring. Major	Minor	*Mimas.* a_1	b_1	l_1-L.	Diff.
1887.	″	″	″	″	″	″	″	°	°
Feb. 2	19·96	·013	18·32	46·01	18·87	31·44	−12·89	259·49	
									1910·22
7	19·86	·018	18·23	45·78	18·84	31·28	12·87	9·71	
									·17
12	19·75	·023	18·13	45·52	18·80	31·10	12·84	119·88	
									·13
17	19·62	·030	18·01	45·23	18·73	30·90	12·80	230·01	
									·08
22	19·48	·035	17·89	44·91	18·64	30·69	12·74	340·09	
									1910·02
27	19·33	·041	17·75	44·57	18·54	30·45	12·67	90·11	
									1909·96
Mar. 4	19·18	·046	17·61	44·21	18·42	30·21	−12·59	200·07	
									·90
9	19·02	·051	17·46	43·84	18·29	29·95	12·50	309·97	
									·85
14	18·85	·055	17·31	43·45	18·14	29·69	12·40	59·82	
									·80
19	18·68	·058	17·15	43·06	17·99	29·42	12·29	169·62	
									·74
24	18·51	·060	17·00	42·67	17·82	29·15	12·18	279·36	
									·69
29	18·34	·061	16·84	42·27	17·65	28·88	12·06	29·05	
									·63
Apr. 3	18·17	·061	16·68	41·88	17·47	28·62	−11·94	138·68	
									1909·59
8	18·00	·061	16·53	41·50	17·29	28·35	11·81	248·27	
									·55
13	17·84	·060	16·38	41·12	17·10	28·10	11·68	357·82	
									·50
18	17·68	·058	16·23	40·75	16·91	27·85	11·55	107·32	
									·46
23	17·53	·055	16·09	40·40	16·72	27·60	11·42	216·78	
									·43
28	17·38	·052	15·95	40·06	16·53	27·37	11·29	326·21	
									·40
May 3	17·24	·048	15·82	39·73	16·34	27·15	−11·17	75·61	
									1909·37
8	17·10	·044	15·69	39·43	16·15	26·94	11·04	184·98	
									·34
13	16·98	·040	15·57	39·14	15·97	26·74	10·91	294·32	
									1909·32
18	16·86	0·035	15·46	38·86	15·79	26·55	−10·79	43·64	

Greenwich Noon.	*Enceladus.* a_2	b_2	l_2-L	Diff.	*Tethys.* a_3	b_3	l_3-L	Diff.
1886.	″	″	°	°	″	″	°	°
Sept. 30	35·62	−13·86	316·50		44·09	−17·15	294·93	
				1313·49				953·29
Oct. 5	35·93	13·94	189·99		44·48	17·25	168·22	
				·53				·33
10	36·26	14·02	63·52		44·88	17·36	41·55	
				·57				·37
15	36·59	14·12	297·09		45·30	17·48	274·92	
				·62				·41
20	36·93	14·23	170·71		45·72	17·61	148·33	
				·67				·46
25	37·27	14·35	44·38		46·14	17·76	21·79	
				·71				·51
30	37·62	14·47	278·09		46·57	17·91	255·30	
				·76				·56
Nov. 4	37·96	−14·60	151·85		46·99	−18·08	128·86	
				1313·80				953·61
9	38·30	14·74	25·65		47·41	18·25	2·47	
				·85				·65
14	38·63	14·89	259·50		47·82	18·43	236·12	
				·89				·69
19	38·95	15·04	133·39		48·22	18·61	109·81	
				·93				·73
24	39·26	15·19	7·32		48·60	18·80	343·54	
				1313·97				·77
29	39·55	15·34	241·29		48·95	18·99	217·31	
				1314·00				·81

Greenwich Noon.		a_2	b_2	l_2-L	Diff.	a_3	b_3	l_3-L	Diff.
1886.		"	"	°		"	"	°	
Dec.	4	39·81	−15·49	115·29		49·28	−19·18	91·12	
					·03				953·84
	9	40·05	15·64	349·32		49·58	19·37	324·96	
					·05				·87
	14	40·26	15·79	223·37		49·84	19·55	198·83	
					·08				·89
	19	40·44	15·93	97·45		50·06	19·72	72·72	
					·09				·91
	24	40·58	16·06	331·54		50·23	19·87	306·63	
					·09				·92
	29	40·69	16·17	205·63		50·37	20·02	180·55	
					·10				·93
1887.									
Jan.	3	40·75	−16·28	79·73		50·45	−20·15	54·48	
					1314·10				953·92
	8	40·78	16·36	313·83		50·48	20·26	288·40	
					·09				·92
	13	40·77	16·44	187·92		50·47	20·35	162·32	
					·07				·91
	18	40·71	16·49	61·99		50·40	20·41	36·23	
					·05				·89
	23	40·62	16·53	296·04		50·29	20·46	270·12	
					1314·02				·86
	28	40·49	16·54	170·06		50·13	20·48	143·98	
					1313·99				·83
Feb.	2	40·33	−16·54	44·05		49·92	−20·47	17·81	
					·95				953·80
	7	40·13	16·52	278·00		49·68	20·44	251·61	
					·90				·76
	12	39·90	16·48	151·90		49·39	20·39	125·37	
					·86				·71
	17	39·65	16·42	25·76		49·08	20·32	359·08	
					·81				·66
	22	39·37	16·34	259·57		48·73	20·23	232·74	
					·76				·62
	27	39·07	16·25	133·33		48·36	20·12	106·36	
					·71				·57
Mar.	4	38·75	−16·15	7·04		47·97	−19·99	339·93	
					1313·65				953·51
	9	38·42	16·03	240·69		47·57	19·84	213·44	
					·60				·46
	14	38·09	15·90	114·29		47·15	19·69	86·90	
					·55				·41
	19	37·75	15·77	347·84		46·72	19·52	320·31	
					·49				·36
	24	37·40	15·62	1·33		46·30	19·34	193·67	
					·44				·31
	29	37·05	15·47	94·77		45·87	19·15	66·98	
					·39				·26
Apr.	3	36·71	−15·31	328·16		45·44	−18·96	300·24	
					1313·34				953·20
	8	36·37	15·15	201·50		45·03	18·76	173·44	
					·30				·17
	13	36·04	14·99	74·80		44·62	18·55	46·61	
					·26				·12
	18	35·72	14·82	308·06		44·22	18·35	279·73	
					·21				·08
	23	35·41	14·66	181·27		43·83	18·14	152·81	
					·18				·04
	28	35·11	14·49	54·45		43·46	17·94	25·85	
					·15				953·01
May	3	34·83	−14·32	287·60		43·11	−17·73	258·86	
					·11				952·98
	8	34·56	14·16	160·71		42·78	17·53	131·84	
					·09				·95
	13	34·30	14·00	33·80		42·46	17·33	4·79	
					1313·06				952·91
	18	34·06	−13·84	266·86		42·17	−17·13	237·70	

Greenwich Noon.		*Dione.*				*Rhea.*			
		a_4	b_4	l_4-L	Diff.	a_5	b_5	l_5-L	Diff.
1886.		″	″	°	°	″	″	°	°
Sept.	30	56·47	−21·97	281·08	657·44	78·86	−30·68	202·87	398·19
Oct.	5	56·97	22·10	218·52	·49	79·56	30·86	241·06	·24
	10	57·48	22·24	156·01	·52	80·28	31·05	279·30	·28
	15	58·01	22·39	93·53	·57	81·01	31·27	317·58	·32
	20	58·55	22·56	31·10	·62	81·77	31·51	355·90	·37
	25	59·10	22·75	328·72	·67	82·53	31·76	34·27	·42
	30	59·64	22·94	266·39	·72	83·29	32·04	72·69	·47
Nov.	4	60·19	−23·15	204·11	657·76	84·05	−32·33	111·16	398·51
	9	60·72	23·37	141·87	·81	84·80	32·64	149·67	·56
	14	61·25	23·60	79·68	·85	85·53	32·96	188·23	·61
	19	61·76	23·84	17·53	·89	86·24	33·29	226·84	·65
	24	62·24	24·08	315·42	·94	86·92	33·63	265·49	·70
	29	62·70	24·32	253·36	657·98	87·55	33·97	304·19	·73
Dec.	4	63·22	−24·56	191·34	658·01	88·14	−34·31	342·92	398·77
	9	63·49	24·80	129·35	·04	88·67	34·64	21·69	·80
	14	63·83	25·03	67·39	·06	89·14	34·96	60·49	·83
	19	64·11	25·25	5·45	·08	89·53	35·26	99·32	·85
	24	64·34	25·45	303·53	·10	89·85	35·55	138·17	·86
	29	64·51	25·64	241·63		90·09	35·81	177·03	·88
1887.									
Jan.	3	64·61	−25·80	179·73	658·11	90·23	−36·04	315·91	398·88
	8	64·66	25·94	117·84	·10	90·29	36·23	254·79	·88
	13	64·64	26·06	55·94	·10	90·26	36·39	293·67	·88
	18	64·55	26·14	354·04	·08	90·15	36·51	332·55	·86
	23	64·41	26·20	292·12	·06	89·95	36·59	11·41	·84
	28	64·20	26·23	230·18	658·03	89·66	36·62	50·25	·81
Feb.	2	63·94	−26·22	168·21	657·99	89·29	−36·62	89·06	398·78
	7	63·62	26·18	106·20	·96	88·85	36·57	127·84	·75
	12	63·26	26·12	44·16	·91	88·34	36·48	166·59	·71
	17	62·86	26·03	342·07	·87	87·78	36·35	205·30	·66
	22	62·41	25·91	279·94	·83	87·16	36·18	243·96	·62
	27	61·94	25·77	217·77	·78	86·50	35·98	282·58	·57
Mar.	4	61·44	−25·60	155·55	657·72	85·80	−35·75	321·15	398·52
	9	60·92	25·42	93·27	·67	85·07	35·49	359·67	·47
	14	60·39	25·21	30·94	·62	84·33	35·21	38·14	·42
	19	59·84	25·00	328·56	·57	83·57	34·91	76·56	·37
	24	59·29	24·77	266·13	·52	82·80	34·59	114·93	·32
	29	58·75	24·53	203·65	·47	82·04	34·25	153·25	·27

Greenwich Noon. 1887.		*Dione.* a_4	b_4	l_4-L	Diff.	*Rhea.* a_5	b_5	l_5-L	Diff.
		″	″	°	°	″	″	°	°
Apr.	3	58·20	−24·28	141·12	657·42	81·28	−33·90	191·52	398·22
	8	57·67	24·02	78·54	·38	80·53	33·55	229·74	·18
	13	57·14	23·76	15·92	·33	79·80	33·18	267·92	·13
	18	56·63	23·50	313·25	·29	79·09	32·82	306·05	·09
	23	56·14	23·24	250·54	·26	78·40	32·45	344·14	·06
	28	55·67	22·97	187·80	·22	77·74	32·08	22·20	398·02
May	3	55·22	−22·71	125·02	·18	77·11	−31·71	60·22	397·98
	8	54·79	22·45	62·20	·15	76·51	31·35	98·20	·95
	13	54·39	22·19	359·35	657·13	75·95	30·99	136·15	397·92
	18	54·01	−21·94	296·48		75·42	−30·63	174·07	

Approximate Greenwich Mean Times of conjunctions of the satellites with the centre of the planet.

"*n*" inferior conjunction with centre, or satellite exactly in the direction of the minor axis of the ring, north, moving from the following to the preceding side.

"*s*" superior conjunction, or satellite south, moving from the preceding to the following side.

Careful observations of the position-angles of the satellites at the times of these conjunctions afford the best means for determining the longitudes of the satellites in their orbits, and are least liable to systematic errors. It is very desirable that they should be regularly made, when they occur at convenient hours.

1886.	h		1886.	h		1886.	h	
Sept. 29	5·5 Di.	n.	Oct. 2	18·4 Mi.	s.	Oct. 5	15·5 En.	s.
	9·5 Te.	n.		21·7 En.	s.	6	0·1 Te.	s.
	17·1 Rh.	n.	3	4·2 Te.	n.		1·8 Di.	s.
30	8·2 Te.	s.		8·1 Di.	s.		8·0 En.	n.
	14·4 Di.	s.		14·2 En.	n.		11·8 Rh.	s.
	20·4 En.	n.		17·0 Mi.	s.		12·8 Mi.	s.
	21·1 Mi.	s.	4	2·8 Te.	s.		22·8 Te.	n.
Oct. 1	6·8 Te.	n.		5·6 Rh.	n.	7	10·7 Di.	n.
	12·9 En.	s.		15·6 Mi.	s.		11·5 Mi.	s.
	19·7 Mi.	s.		17·0 Di.	n.		16·9 En.	n.
	23·3 Di.	n.		23·1 En.	n.		21·5 Te.	s.
	23·3 Rh.	s.	5	1·5 Te.	n.	8	9·3 En.	s.
2	5·5 Te.	s.		14·2 Mi.	s.		18 1 Rh.	n.

1886.	h		
Oct. 8	19·5	Di.	s.
	20·1	Te.	n.
	21·4	Mi.	n.
9	18·2	En.	s.
	18·8	Te.	s.
	20·0	Mi.	n.
10	4·4	Di.	n.
	10·6	En.	n.
	17·4	Te.	n.
	18·6	Mi.	n.
11	0·3	Rh.	s.
	13·2	Di.	s.
	16·1	Te.	s.
	17·2	Mi.	n.
	19·5	En.	n.
12	12·0	En.	s.
	14·7	Te.	n.
	15·9	Mi.	n.
	22·1	Di.	n.
13	6·5	Rh.	n.
	13·4	Te.	s.
	14·5	Mi.	n.
	20·9	En.	s.
14	6·9	Di.	s.
	12·0	Te.	n.
	13·1	Mi.	n.
	13·3	En.	n.
15	10·7	Te.	s.
	11·7	Mi.	n.
	12·8	Rh.	s.
	15·8	Di.	n.
	22·2	En.	n.
16	9·4	Te.	n.
	14·6	En.	s.
	21·7	Mi.	s.
17	0·6	Di.	s.
	8·0	Te.	s.
	19·0	Rh.	n.
18	6·7	Te.	n.

1886.	h		
Oct. 18	9·5	Di.	n.
	16·0	En.	n.
19	5·3	Te.	s.
	18·3	Di.	s.
20	1·2	Rh.	s.
	4·0	Te.	n.
	17·3	En.	s.
21	2·6	Te.	s.
	3·2	Di.	n.
	9·7	En.	n.
	14·7	Mi.	s.
22	1·3	Te.	n.
	7·5	Rh.	n.
	12·0	Di.	s.
	13·4	Mi.	s.
	18·6	En.	n.
	23·9	Te.	s.
23	11·1	En.	s.
	12·0	Mi.	s.
	20·9	Di.	n.
	22·6	Te.	n.
24	10·6	Mi.	s.
	13·7	Rh.	s.
	19·9	En.	s.
	21·3	Te.	s.
25	5·7	Di.	s.
	12·4	En.	n.
	19·9	Te.	n.
	20·5	Mi.	n.
26	14·5	Di.	n.
	18·6	Te.	s.
	19·1	Mi.	n.
	19·9	Rh.	n.
	21·3	En.	n.
27	13·7	En.	s.
	17·2	Te.	n.
	17·8	Mi.	n.
	23·4	Di.	s.
28	15·9	Te.	s.

1886.	h		
Oct. 28	16·4	Mi.	n.
	22·6	Te.	s.
29	2·1	Rh.	s.
	8·2	Di.	n.
	14·5	Te.	n.
	15·0	Mi.	n.
	15·0	En.	n.
30	13·2	Te.	s.
	13·6	Mi.	n.
	17·1	Di.	s.
	23·9	En.	n.
31	8·3	Rh.	n.
	11·8	Te.	n.
	12·2	Mi.	n.
	16·4	En.	s.
Nov. 1	1·9	Di.	n.
	8·8	En.	n.
	10·5	Te.	s.
	10·8	Mi.	n.
2	9·1	Te.	n.
	10·8	Di.	s.
	14·5	Rh.	s.
	17·7	En.	n.
	20·8	Mi.	s.
3	7·8	Te.	s.
	10·1	En.	s.
	19·4	Mi.	s.
	19·6	Di.	n.
4	6·4	Te.	n.
	18·0	Mi.	s.
	19·0	En.	s.
	20·7	Rh.	n.
5	4·4	Di.	s.
	5·1	Te.	s.
	11·4	En.	n.
	16·6	Mi.	s.
6	3·7	Te.	n.
	13·3	Di.	n.
	15·2	Mi.	s.

1886.	h		
Nov. 6	20·3	En.	n.
7	2·4	Te.	s.
	2·9	Rh.	s.
	12·8	En.	n.
	13·9	Mi.	s.
	22·1	Di.	s.
8	1·0	Te.	n.
	12·5	Mi.	s.
	21·7	En.	s.
	23·7	Te.	s.
9	6·9	Di.	n.
	9·1	Rh.	n.
	11·1	Mi.	s.
	14·1	En.	n.
	22·3	Te.	n.
10	15·8	Di.	s.
	21·0	Te.	s.
	21·0	Mi.	n.
	23·0	En.	n.
11	15·3	Rh.	s.
	15·4	En.	s.
	19·6	Te.	n.
	19·6	Mi.	n.
12	0·6	Di.	n.
	7·9	En.	n.
	18·2	Mi.	n.
	18·3	Te.	s.
13	9·5	Di.	s.
	16·7	En.	n.
	16·9	Mi.	n.
	16·9	Te.	n.
	21·5	Rh.	n.
14	15·6	Te.	s.
	18·3	Di.	n.
15	14·2	Te.	n.
16	3·1	Di.	s.
	3·7	Rh.	s.
	12·9	Te.	s.
17	11·3	Mi.	n.
	11·5	Te.	n.

1886.	h		
Nov. 17	12·0	Di.	n.
	19·4	En.	n.
18	9·9	Rh.	n.
	10·2	Te.	s.
	11·8	En.	s.
	20·8	Di.	s.
	21·2	Mi.	s.
19	8·8	Te.	n.
	19·8	Mi.	s.
	20·7	En.	s.
20	5·6	Di.	n.
	7·5	Te.	s.
	13·1	En.	n.
	16·1	Rh.	s.
	18·5	Mi.	s.
21	6·1	Te.	n.
	14·5	Di.	s.
	17·1	Mi.	s.
	22·0	En.	n.
22	4·8	Te.	s.
	14·4	En.	s.
	15·7	Mi.	s.
	22·3	Rh.	n.
	23·3	Di.	n.
23	3·4	Te.	n.
	6·9	En.	n.
	14·3	Mi.	s.
24	2·1	Te.	s.
	8·1	Di.	s.
	12·9	Mi.	s.
	15·8	En.	n.
25	0·7	Te.	n.
	4·4	Rh.	s.
	8·2	En.	s.
	11·5	Mi.	s.
	17·0	Di.	n.
	23·4	Te.	s.
26	10·1	Mi.	s.
	17·1	En.	s.
	22·0	Te.	n.

1886.	h		
Nov. 27	1·8	Di.	s.
	9·5	En.	n.
	10·6	Rh.	n.
	20·1	Mi.	n.
	20·7	Te.	s.
28	10·6	Di.	n.
	18·4	En.	n.
	18·7	Mi.	n.
	19·3	Te.	n.
29	10·8	En.	s.
	16·8	Rh.	s.
	17·3	Mi.	n.
	17·9	Te.	s.
	19·4	Di.	s.
30	15·9	Mi.	n.
	16·6	Te.	n.
	19·7	En.	s.
Dec. 1	4·3	Di.	n.
	12·2	En.	n.
	14·5	Mi.	n.
	15·2	Te.	s.
	23·0	Rh.	n.
2	13·1	Di.	s.
	13·1	Mi.	n.
	13·9	Te.	n.
	21·0	En.	n.
3	11·7	Mi.	n.
	12·5	Te.	s.
	13·5	En.	s.
	21·9	Di.	n.
4	5·1	Rh.	s.
	5·9	En.	n.
	10·4	Mi.	n.
	11·2	Te.	n.
5	6·8	Di.	s.
	9·8	Te.	s.
	14·8	En.	n.
	20·3	Mi.	s.
6	7·2	En.	s.
	8·5	Te.	n.

1886.	h		
Dec. 6	11·3	Rh.	n.
	15·6	Di.	n.
	18·9	Mi.	s.
7	7·1	Te.	s.
	16·1	En.	s.
	17·5	Mi.	s.
8	0·4	Di.	s.
	5·7	Te.	n.
	8·5	En.	n.
	16·1	Mi.	s.
	17·5	Rh.	s.
9	4·4	Te.	s.
	9·2	Di.	n.
	14·7	Mi.	s.
	17·4	En.	n.
10	3·0	Te.	n.
	9·8	En.	s.
	13·3	Mi.	s.
	18·1	Di.	s.
	23·6	Rh.	n.
11	1·7	Te.	s.
	12·0	Mi.	s.
	18·7	En.	s.
12	0·3	Te.	n.
	2·9	Di.	n.
	23·0	Te.	s.
13	5·8	Rh.	s.
	11·7	Di.	s.
	21·6	Te.	n.
14	12·5	En.	s.
	20·3	Te.	s.
	20·5	Di.	n.
15	11·9	Rh.	n.
	17·7	Mi.	n.
	18·9	Te.	n.
	21·3	En.	s.
16	5·4	Di.	s.
	13·8	En.	n.
	16·3	Mi.	n.
	17·6	Te.	s.

1886.	h		
Dec. 17	6·2	En.	s.
	14·2	Di.	n.
	14·9	Mi.	n.
	16·2	Te.	n.
	18·1	Rh.	s.
18	13·6	Mi.	n.
	14·8	Te.	s.
	15·1	En.	s.
	23·0	Di.	s.
19	7·5	En.	n.
	12·2	Mi.	n.
	13·5	Te.	n.
20	0·3	Rh.	n.
	7·8	Di.	n.
	10·8	Mi.	n.
	12·1	Te.	s.
	16·4	En.	n.
21	8·8	En.	s.
	9·4	Mi.	n.
	10·8	Te.	n.
	16·6	Di.	s.
22	6·4	Rh.	s.
	8·0	Mi.	n.
	9·4	Te.	s.
	17·7	En.	s.
	19·3	Mi.	s.
23	1·5	Di.	n.
	8·1	Te.	n.
	10·2	En.	n.
	17·9	Mi.	s.
24	6·7	Te.	s.
	10·3	Di.	s.
	12·6	Rh.	n.
	16·5	Mi.	s.
	19·0	En.	n.
25	5·4	Te.	n.
	11·5	En.	s.
	15·2	Mi.	s.
	19·1	Di.	n.
26	4·0	Te.	s.

1886.	h		
Dec. 26	13·8	Mi.	s.
	18·7	Rh.	s.
	20·3	En.	s.
27	2·6	Te.	n.
	3·9	Di.	s.
	12·4	Mi.	s.
	12·8	En.	n.
28	1·3	Te.	s.
	5·2	En.	s.
	11·0	Mi.	s.
	12·8	Di.	n.
	23·9	Te.	n.
29	0·9	Rh.	n.
	9·6	Mi.	s.
	14·1	En.	s.
	21·6	Di.	s.
	22·6	Te.	s.
30	6·5	En.	n.
	8·2	Mi.	s.
	21·2	Te.	n.
31	6·4	Di.	n.
	7·0	Rh.	s.
	15·4	En.	n.
	18·1	Mi.	n.
	19·9	Te.	s.
1887.			
Jan. 1	7·8	En.	s.
	15·2	Di.	s.
	16·7	Mi.	n.
	18·5	Te.	n.
2	13·2	Rh.	n.
	15·4	Mi.	n.
	16·7	En.	s.
	17·1	Te.	s.
3	0·0	Di.	n.
	9·2	En.	n.
	14·0	Mi.	n.
	16·0	Te.	n.
4	8·9	Di.	s.
	12·6	Mi.	n.

1887.	h		
Jan. 4	14·4	Te.	s.
	18·0	En.	n.
	19·3	Rh.	s.
5	10·5	En.	s.
	11·2	Mi.	n.
	13·1	Te.	n.
	17·7	Di.	n.
6	9·8	Mi.	n.
	11·7	Te.	s.
	19·3	En.	s.
7	1·5	Rh.	n.
	2·5	Di.	s.
	8·4	Mi.	n.
	10·4	Te.	n.
	11·8	En.	n.
8	9·0	Te.	s.
	11·3	Di.	n.
9	7·6	Rh.	s.
	7·6	Te.	n.
	20·2	Di.	s.
10	6·3	Te.	s.
11	4·9	Te.	n.
	5·0	Di.	n.
	13·8	Rh.	n.
12	3·6	Te.	s.
	13·8	Di.	s.
13	2·2	Te.	n.
	11·3	Mi.	s.
	15·7	En.	s.
	19·9	Rh.	s.
14	0·6	Di.	n.
	0·9	Te.	s.
	8·1	En.	n.
	9·9	Mi.	s.
	23·5	Te.	n.
15	8·6	Mi.	s.
	9·4	Di.	s.
	17·0	En.	n.
	22·2	Te.	s.

1887.	h		
Jan. 16	2·0	Rh.	n.
	7·3	Mi.	s.
	9·5	En.	s.
	18·3	Di.	n.
	20·8	Te.	n.
17	17·2	Mi.	n.
	18·3	En.	s.
	19·4	Te.	s.
18	3·1	Di.	s.
	8·3	Rh.	s.
	10·8	En.	n.
	15·8	Mi.	n.
	18·1	Te.	n.
19	11·9	Di.	n.
	14·4	Mi	n.
	16·7	Te.	s.
	19·6	En.	n.
20	12·1	En.	s.
	13·0	Mi.	n.
	14·4	Rh.	n.
	15·4	Te.	n.
	20·7	Di.	s.
21	4·5	En.	n.
	11·6	Mi.	n.
	14·0	Te.	s.
22	5·6	Di.	n.
	10·2	Mi.	n.
	12·7	Te.	n.
	13·4	En.	n.
	20·6	Rh.	s.
23	5·8	En.	s.
	8·9	Mi.	n.
	11·3	Te.	s.
	14·4	Di.	s.
24	7·5	Mi.	n.
	9·9	Te.	n.
	14·7	En.	s.
	23·2	Di.	n.
25	2·7	Rh.	n.

1887.	h		
Jan. 25	6·1	Mi.	n.
	7·2	En.	n.
	8·6	Te.	s.
26	7·2	Te.	n.
	8·0	Di.	s.
	16·0	Mi.	s.
	16·0	En.	n.
27	5·9	Te.	s.
	8·5	En.	s.
	8·9	Rh.	s.
	14·6	Mi.	s.
	14·8	Di.	n.
28	4·5	Te.	n.
	13·2	Mi.	s.
	17·3	En.	s.
	23·7	Di.	s.
29	3·2	Te.	s.
	9·8	En.	n.
	11·9	Mi.	s.
	15·0	Rh.	n.
30	1·8	Te.	n.
	8.5	Di.	n.
	10·5	Mi.	s.
	18·7	En.	n.
31	0·5	Te.	s.
	9·1	Mi.	s.
	11·1	En.	s.
	17·3	Di.	s.
	21·2	Rh.	s.
	23·1	Te.	n.
Feb. 1	3·5	En.	n.
	7·7	Mi.	s.
	21·8	Te.	s.
2	2·2	Di.	n.
	6·3	Mi.	s.
	12·4	En.	n.
	17·6	Mi.	n.
	20·4	Te.	n.
3	3·4	Rh.	n.

1887.	h		
Feb. 3	4·9	En.	s.
	4·9	Mi.	s.
	11·0	Di.	s.
	16·2	Mi.	n.
	19·0	Te.	s.
4	13·7	En.	s.
	14·8	Mi.	n.
	17·7	Te.	n.
	19·8	Di.	n.
5	9·5	Rh.	s.
	16·3	Te.	s.
6	4·6	Di.	s.
	15·0	Te.	n.
7	7·5	En.	s.
	10·7	Mi.	n.
	13·5	Di.	n.
	13·6	Te.	s.
	15·7	Rh.	n.
8	9·3	Mi.	n.
	12·3	Te.	n.
	16·4	En.	s.
	20·6	Mi.	s.
	22·3	Di.	s.
9	7·9	Mi.	n.
	8·8	En.	n.
	10·9	Te.	s.
	21·9	Rh.	s.
10	6·5	Mi.	n.
	7·1	Di.	n.
	9·6	Te.	n.
	17·7	En.	n.
	17·8	Mi.	s.
11	8·2	Te.	s.
	10·1	En.	s.
	15·9	Di.	s.
	16·5	Mi.	s.
12	4·0	Rh.	n.
	6·9	Te.	n.
	15·1	Mi.	s.

1887.	h		
Feb. 12	19·0	En.	s.
13	0·8	Di.	n.
	5·5	Te.	s.
	11·4	En.	n.
	13·7	Mi.	s.
14	3·9	En.	s.
	4·2	Te.	n.
	9·6	Di.	s.
	10·2	Rh.	s.
	12·3	Mi.	s.
15	2·8	Te.	s.
	10·9	Mi.	s.
	12·8	En.	s.
	18·4	Di.	n.
16	1·5	Te.	n.
	5·2	En.	n.
	9·5	Mi.	s.
	16·4	Rh.	n.
17	0·1	Te.	s.
	3·3	Di.	s.
	8·2	Mi.	s.
	14·1	En.	n.
	22·8	Te.	n.
18	6·5	En.	s.
	6·8	Mi.	s.
	12·1	Di.	n.
	18·1	Mi.	n.
	21·4	Te.	s.
	22·6	Rh.	s.
19	15·4	En.	s.
	16·7	Mi.	n.
	20·1	Te.	n.
	20·9	Di.	s.
20	7·8	En.	n.
	15·3	Mi.	n.
	18·7	Te.	s.
21	4·7	Rh.	n.
	5·8	Di.	n.
	13·9	Mi.	n.

1887.	h		
Feb. 21	16·7	En.	n.
	17·4	Te.	n.
22	9·2	En.	s.
	12·6	Mi.	n.
	14·6	Di.	s.
	16·0	Te.	s.
23	10·9	Rh.	s.
	11·2	Mi.	n.
	14·7	Te.	n.
	18·1	En.	s.
	23·4	Di.	n.
24	9·8	Mi.	n.
	10·5	En.	n.
	13·3	Te.	s.
25	2·9	En.	s.
	8·3	Di.	s.
	8·4	Mi.	n.
	12·0	Te.	n.
	17·1	Rh.	n.
26	7·0	Mi.	n.
	10·6	Te.	s.
	11·8	En.	s.
	17·1	Di.	n.
27	4·3	En.	n.
	5·6	Mi.	n.
	9·3	Te.	n.
	17·0	Mi.	s.
	23·3	Rh.	s.
28	1·9	Di.	s.
	7·9	Te.	s.
	13·1	En.	n.
	15·6	Mi.	s.
March 1	5·6	En.	s.
	6·6	Te.	n.
	10·8	Di.	n.
	14·2	Mi.	s.
2	5·2	Te.	s.
	5·5	Rh.	n.
	12·8	Mi.	s.

1887.	h		
March 2	14·5	En.	s.
	19·6	Di.	s.
3	3·9	Te.	n.
	6·9	En.	n.
	11·4	Mi.	s.
4	2·5	Te.	s.
	4·5	Di.	n.
	11·7	Rh.	s.
5	1·2	Te.	n.
	13·3	Di.	s.
	23·8	Te.	s.
6	17·9	Rh.	n.
	22·1	Di.	n.
	22·5	Te.	n.
7	9·6	En.	n.
	17·2	Mi.	n.
	21·1	Te.	s.
8	7·0	Di.	s.
	15·8	Mi.	n.
	18·5	En.	n.
	19·8	Te.	n.
9	0·1	Rh.	s.
	10·9	En.	s.
	14·4	Mi.	n.
	15·8	Di.	n.
	18·4	Te.	s.
10	3·3	En.	n.
	13·1	Mi.	n.
	17·1	Te.	n.
11	0·7	Di.	s.
	6·3	Rh.	n.
	11·7	Mi.	n.
	12·2	En.	n.
	15·7	Te.	s.
12	4·7	En.	s.
	9·5	Di.	n.
	10·3	Mi.	n.
	14·4	Te.	n.
13	8·9	Mi.	n.

1887.	h		
March 13	12·5	Rh.	s.
	13·1	Te.	s.
	13·6	En.	s.
	18·3	Di.	s.
14	6·0	En.	n.
	7·5	Mi.	n.
	11·7	Te.	n.
15	3·2	Di.	n.
	6·2	Mi.	n.
	10·4	Te.	s.
	14·9	En.	n.
	18·7	Rh.	n.
16	4·8	Mi.	n.
	7·3	En.	s.
	9·0	Te.	n.
	12·0	Di.	s.
	16·1	Mi.	s.
17	7·7	Te.	s.
	14·7	Mi.	s.
	16·2	En.	s.
	20·9	Di.	n.
18	0·9	Rh.	s.
	6·3	Te.	n.
	8·7	En.	n.
	13·3	Mi.	s.
19	5·0	Te.	s.
	5·7	Di.	s.
	12·0	Mi.	s.
	17·6	En.	n.
20	3·6	Te.	n.
	7·2	Rh.	n.
	10·0	En.	s.
	10·6	Mi.	s.
	14·6	Di.	n.
21	2·3	Te.	s.
	2·4	En.	n.
	9·2	Mi.	s.
	23·4	Di.	s.
22	1·0	Te.	n.

1887.	h		
March 22	7·8	Mi.	s.
	11·3	En.	n.
	13·4	Rh.	s.
	23·6	Te.	s.
23	3·8	En.	s.
	6·4	Mi.	s.
	8·3	Di.	n.
	22·3	Te.	n.
24	12·7	En.	s.
	16·4	Mi.	n.
	17·1	Di.	s.
	19·6	Rh.	n.
	20·9	Te.	s.
25	5·1	En.	n.
	15·0	Mi.	n.
	19·6	Te.	n.
26	2·0	Di.	n.
	13·6	Mi.	n.
	14·0	En.	n.
	18·2	Te.	s.
27	1·8	Rh.	s.
	6·4	En.	s.
	10·8	Di.	s.
	12·2	Mi.	n.
	16·9	Te.	n.
28	10·9	Mi.	n.
	15·3	En.	s.
	15·6	Te.	s.
	19·7	Di.	n.
29	7·8	En.	n.
	8·1	Rh.	n.
	9·5	Mi.	n.
	14·2	Te.	n.
30	0·2	En.	s.
	4·5	Di.	s.
	8·1	Mi.	n.
	12·9	Te.	s.
31	6·7	Mi.	n.
	9·1	En.	s.

1887.	h		
March 31	11·5	Te.	n.
	13·4	Di.	n.
	14·3	Rh.	s.
April 1	10·2	Te.	s.
	22·2	Di.	s.
2	8·9	Te.	n.
	10·5	En.	n.
	20·5	Rh.	n.
3	2·9	En.	s.
	7·1	Di.	n.
	7·5	Te.	s.
	13·9	Mi.	s.
4	6·2	Te.	n.
	11·8	En.	s.
	12·5	Mi.	s.
	16·0	Di.	s.
5	2·9	Rh.	s.
	4·2	En.	n.
	4·8	Te.	s.
	11·1	Mi.	s.
6	0·8	Di.	n.
	3·5	Te.	n.
	9·8	Mi.	s.
	13·1	En.	n.
7	2·2	Te.	s.
	5·6	En.	s.
	8·4	Mi.	s.
	9·0	Rh.	n.
	9·7	Di.	s.
8	0·8	Te.	n.
	7·0	Mi.	s.
	14·5	En.	s.
	18·5	Di.	n.
	23·5	Te.	s.
9	5·6	Mi.	n.
	6·9	En.	s.
	15·2	Rh.	s.
	22·1	Te	n.
10	3·4	Di.	s.

1887.	h		
April 10	4·3	Mi.	s.
	15·8	En.	n.
	20·8	Te.	s.
11	8·3	En.	s.
	12·2	Di.	n.
	14·2	Mi.	n.
	19·5	Te.	n.
	21·5	Rh.	n.
12	12·8	Mi.	n.
	17·2	En.	s.
	18·1	Te.	s.
	21·1	Di.	s.
13	9·6	En.	n.
	11·4	Mi.	n.
	16·8	Te.	n.
14	2·1	En.	s.
	3·8	Rh.	s.
	6·0	Di.	n.
	10·1	Mi.	n.
	15·5	Te.	s.
15	8·7	Mi.	n.
	10·9	En.	s.
	14·1	Te.	n.
	14·8	Di.	s.
16	3·4	En.	n.
	7·3	Mi.	n.
	10·0	Rh.	n.
	12·8	Te.	s.
	23·7	Di.	n.
17	11·4	Te.	n.
	12·3	En.	n.
18	4·7	En.	s.
	8·5	Di.	s.
	10·1	Te.	s.
	16·3	Rh.	s.
19	8·8	Te.	n.
	13·6	En.	s.
	14·5	Mi.	s.
	17·4	Di.	n.

1887.	h		
April 20	6·1	En.	n.
	7·4	Te.	s.
	13·1	Mi.	s.
	22·5	Rh.	n.
21	2·3	Di.	s.
	6·1	Te.	n.
	11·7	Mi.	s.
	15·0	En.	n.
22	4·8	Te.	s.
	7·4	En.	s.
	10·4	Mi.	s.
	11·1	Di.	n.
23	3·4	Te.	n.
	4·8	Rh.	s.
	20·0	Di.	s.
24	2·1	Te.	s.
25	0·7	Te.	n.
	4·8	Di.	n.
	11·0	Rh.	n.
	23·4	Te.	s.
26	13·7	Di.	s.
	22·1	Te.	n.
27	17·3	Rh.	s.
	20·7	Te.	s.
	22·6	Di.	n.
28	19·4	Te.	n.
29	7·4	Di.	s.
	18·1	Te.	s.
	23·4	Rh.	n.
30	16·3	Di.	n.
	16·7	Te.	n.
May 1	15·4	Te.	s.
2	1·2	Di.	s.
	5·8	Rh.	s.
	14·1	Te.	n.
3	10·0	Di.	n.
	12·7	Te.	s.
4	11·4	Te.	n.
	12·1	Rh.	n.

1887.	h		1887.	h		1887.	h	
May 4	18·9	Di. s.	May 7	7·4	Te. s.	May 9	0·7	Rh. n.
5	10·1	Te. s.		12·4	Di. s.		4·7	Te. s.
6	3·8	Di. n.	8	6·1	Te. n.	10	3·4	Te. n.
	8·7	Te. n.		21·5	Di. n.		6·4	Di. s.
	18·4	Rh. s.						

By means of this list of conjunctions, approximate values of the coördinates x and y, expressed in semi-diameters of the planet's equator, may be easily found for any other time t in the following little table, the argument of which is the interval τ between the time t and the time of the next preceding or following conjunction. The satellites are on the same side, north or south, of the major axis of the ring as at the time of the next conjunction, and, as they move in the direction of *de*creasing position-angles, they are in the interval between a "n" and a "s" conjunction on the preceding side, and between a "s" and a "n" conjunction on the following side of the direction of the minor axis of the ring. They are at their greatest elongations *Mi.* $5^h{\cdot}7$, *En.* $8^h{\cdot}2$, *Te.* $11^h{\cdot}3$, *Di.* $16^h{\cdot}4$, *Rh.* $27^h{\cdot}1$, before or after the times of conjunction.

τ	*Mimas.*		*Enceladus.*		*Tethys.*		*Dione.*		*Rhea.*	
h	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5
0	0·0	1·3	0·0	1·6	0·0	2·0	0·0	2·6	0·0	3·6
1	0·9	1·2	0·8	1·6	0·7	2·0	0·6	2·5	0·5	3·6
2	1·7	1·1	1·5	1·5	1·4	1·9	1·2	2·5	1·0	3·6
3	2·3	0·8	2·2	1·4	2·0	1·8	1·8	2·5	1·5	3·5
4	2·8	0·6	2·8	1·2	2·6	1·7	2·4	2·4	2·0	3·5
5	3·1	0·2	3·3	0·9	3·2	1·5	2·9	2·3	2·5	3·4
6	3·1	0·1	3·7	0·7	3·7	1·3	3·5	2·1	3·0	3·4
7	2·9	0·5	3·9	0·4	4·1	1·1	4·0	2·0	3·5	3·3
8	2·5	0·8	4·0	0·1	4·5	0·9	4·4	1·8	4·0	3·2
9	1·8	1·0	4·0	0·2	4·7	0·7	4·8	1·7	4·5	3·1
10	1·1	1·2	3·8	0·5	4·9	0·4	5·2	1·5	4·9	3·0
11					5·0	0·1	5·6	1·3	5·3	2·9
12							5·8	1·1	5·7	2·7
13							6·1	0·8	6·1	2·6
14							6·2	0·6	6·5	2·5
15							6·3	0·3	6·8	2·3
16							6·4	0·1	7·1	2·1
20									8·2	1·4
27·1									8·9	0·0

Observers who are desirous to follow the motions of the satellites will do well to lay down the data of the table graphically

on a sufficiently large scale, so that, by marking the corresponding times for the night of observing, they may get information about the places of the satellites at a glance.

Differences of Right Ascension and Declination between the three outer Satellites and the Centre of Saturn.

Greenwich Noon. 1886.		*Titan.*		*Hyperion.*		*Iapetus.*	
		α_6–A.	δ_6–D.	α_7–A.	δ_7–D.	α_8–A.	δ_8–D.
		s	"	s	"	s	"
Sept.	30	− 7·36	−71·0	−13·61	+ 6·5	−26·71	−103·8
Oct.	1	− 2·82	−74·8	−14·96	− 21·3	24·63	104·9
	2	+ 2·13	−68·2	−14·80	− 47·0	−22·40	−105·4
	3	+ 6·80	−51·9	−13·28	− 68·5	20·02	105·2
	4	+10·49	−28·0	−10·68	− 84·4	−17·50	−104·4
	5	+12·64	+ 0·0	− 7·26	− 94·1	14·86	102·9
	6	+12·83	+28·2	− 3·33	− 97·3	−12·13	−100·8
	7	+11·00	+51·9	+ 0·84	− 94·2	9·31	98·1
	8	+ 7·35	+67·1	+ 4·96	− 85·2	− 6·42	− 94·8
	9	+ 2·48	+71·3	+ 8·80	− 70·9	3·48	91·0
	10	− 2·83	+63·5	+12·10	− 52·1	− 0·51	− 86·6
	11	− 7·70	+45·3	+14·64	− 29·9	+ 2·48	81·7
	12	−11·34	+19·7	+16·21	− 5·8	+ 5·46	− 76·3
	13	−13·23	− 9·1	+16·61	+ 18·9	8·42	70·5
	14	−13·14	−36·6	+15·70	+ 42·2	+11·34	− 64·2
	15	−11·12	−58·8	+13·42	+ 61·8	14·21	57·6
	16	− 7·50	−72·7	+ 9·84	+ 75·4	+17·01	− 50·6
	17	− 2·81	−76·4	+ 5·19	+ 81·0	19·73	43·3
	18	+ 2·30	−69·4	− 0·08	+ 77·1	+22·34	− 35·8
	19	+ 7·09	−52·3	− 5·35	+ 63·6	24·83	28·0
	20	+10·87	−27·8	− 9·98	+ 43·1	+27·19	− 20·0
	21	+13·03	+ 1·0	−13·41	+ 15·4	29·40	11·9
	22	+13·19	+29·8	−15·32	− 13·3	+31·46	− 3·8
	23	+11·23	+53·8	−15·65	− 40·6	33·34	− 4·4
	24	+ 7·43	+69·1	−14·51	− 64·5	+35·04	+ 12·6
	25	+ 2·37	+73·0	−12·16	− 82·7	36·54	20·8
	26	− 3·10	+64·6	− 8·86	− 94·9	+37·84	+ 28·9
	27	− 8·08	+45·5	− 4·92	−100·4	38·93	36·8
	28	−11·78	+19·1	− 0·63	− 99·3	+39·80	+ 44·6
	29	−13·66	−10·5	+ 3·71	− 91·9	40·44	52·1
	30	−13·49	−38·6	+ 7·84	− 78·7	+40·85	+ 59·3
	31	−11·33	−61·2	+11·49	− 60·6	41·02	66·3

Greenwich Noon. 1886.		*Titan.* $\alpha_6-A.$	$\delta_6-D.$	*Hyperion.* $\alpha_7-A.$	$\delta_7-D.$	*Iapetus.* $\alpha_8-A.$	$\delta_8-D.$
		s	"	s	"	s	"
Nov.	1	− 7·54	−75·1	+14·43	− 38·6	+40·96	+ 72·9
	2	− 2·67	−78·4	+16·41	− 13·9	40·65	79·1
	3	+ 2·59	−70·7	+17·23	+ 11·8	+40·10	+ 84·8
	4	+ 7·50	−52·9	+16·71	+ 36·7	39·31	90·1
	5	+11·33	−27·2	+14·76	+ 58·5	+38·29	+ 94·9
	6	+13·47	+ 2·6	+11·40	+ 74·8	37·04	99·1
	7	+13·52	+32·0	+ 6·83	+ 83·2	+35·55	+102·7
	8	+11·40	+56·5	+ 1·46	+ 82·0	33·85	105·8
	9	+ 7·40	+71·7	− 4·11	+ 70·7	+31·93	+108·3
	10	+ 2·14	+75·0	− 9·18	+ 50·5	29·80	110·1
	11	− 3·49	+65·7	−13·13	+ 23·9	+27·49	+111·2
	12	− 8·57	+45·5	−15·58	− 5·7	24·99	111·6
	13	−12·29	+17·9	−16·37	− 34·8	+22·32	+111·4
	14	−14·11	−12·6	−15·59	− 60·7	19·51	110·4
	15	−13·79	−41·4	−13·46	− 81·5	+16·56	+108·8
	16	−11·45	−64·2	−10·26	− 96·0	13·49	106·4
	17	− 7·46	−78·0	− 6·31	−103·7	+10·32	+103·4
	18	− 2·40	−80·7	− 1·92	−104·5	7·08	99·6
	19	+ 3·02	−72·0	+ 2·61	− 98·5	+ 3·77	+ 95·2
	20	+ 8·02	−53·0	+ 6·99	− 86·4	+ 0·43	+ 90·2

The rest will be published in the Supplementary Number of the *Monthly Notices*.

In order that the errors of the predicted positions of *Hyperion* may be kept within sufficiently narrow limits, it is essential that the latest observed positions should be available. Observers will oblige by timely publication or communication of the positions resulting from their measurements.